Clustering Instability in Dissipative Gases

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It is shown that a gas composed of inelastically colliding particles is unstable to the formation of high density clusters. A possible physical mechanism underlying this instability is proposed. A theoretical analysis, based on the Jenkins-Richman equations, as well as a numerical simulation of the dynamics of an unforced system of hard disks in a periodic rectangular enclosure, renders support to the proposed mechanism. In particular, a simple formula for the characteristic intercluster distance is derived and found to be in agreement with the numerical results. Applications to granular systems of engineering interest as well as to astrophysics are briefly outlined.

The difference between a gas of particles colliding inelastically and a regular gas is in the inherent dissipative nature of the elementary collision process. Even if the degree of inelasticity is minute, its mere existence can give rise to a significantly different physics than one encounters in the statistical mechanics of regular gases. A large variety of excited granular systems [1,2] belong to this class of "granular gases."

One of the interesting properties of granular gases is their tendency to form dense clusters of particles [3]. This fact can be intuitively understood as follows. In a region in which the density is increased due to a fluctuation (without a change in the granular temperature), the frequency of collisions is larger then in neighboring, less dense regions. Since the collisions are inelastic, by assumption, the granular temperature in the dense region will decrease, causing a decrease in pressure in that region. The resulting pressure gradient will lead to a migration of particles into the dense region, thus further increasing its density and decreasing its pressure. Thus, once commenced, a density fluctuation will lead to the formation of a cluster, provided the (hydrodynamic) mechanisms that may disperse the agglomeration of particles are slower than the clustering process.

A simple system which can serve to illustrate the clustering phenomenon is a granular gas composed of identical rigid spherical particles (disks in two dimensions) whose collisions are characterized by a fixed coefficient of normal restitution. Consider such a system (say, of infinite extent) prepared in an initial state of uniform macroscopic density (as measured on scales larger than the mean free path), uniform granular temperature, and a vanishing (macroscopic) velocity field. Naive intuition may suggest that the system would remain uniform in space, its granular temperature decaying due to the inelasticity of the collisions. However, since mass and momentum are conserved quantities, the density and

momentum density are hydrodynamic variables [4,5] and their decay rate is slower the larger the typical (wave)-length of the fluctuations. Thus a shear mode *fluctuation* of a large enough wavelength will decay at a much slower pace than that characterizing the (locally determined) rate of (collisional) cooling. The viscous heating in relatively high shear regions will increase the pressure there and cause a motion of particles towards the lower pressure (lower shear as well) regions. Once a density nonuniformity is created, the process of clustering will proceed as explained above.

These effects can be demonstrated by an analysis of the following equations of motion [6,7] for (three-dimensional) granular fluids:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla \cdot \mathbf{P}, \tag{1}$$

$$\frac{3}{2}\rho \frac{DT}{Dt} = -\nabla \cdot \mathbf{Q} - \mathrm{tr}(\mathbf{PD}) - \gamma, \tag{2}$$

$$\dot{\rho} = -\nabla \cdot (\rho \mathbf{v}), \tag{3}$$

where D/Dt is the material derivative, ρ the mass density, T the granular temperature, $\mathbf{v}$ the velocity field, $\mathbf{D}$ the symmetrized velocity gradient tensor, $\mathbf{P}$ the stress tensor, $\mathbf{Q}$ the heat flux, and γ represents the rate of energy loss due to inelastic collisions. These equations of motion, which are of quite general nature, must be supplemented by constitutive relations. Here we employ the constitutive relations derived by Jenkins and Richman [6] by using a kinetic approach. While one may argue about the validity of *any* constitutive relation for granular fluids, we believe on the basis of results of molecular dynamic (MD) simulations [1,8] that these constitutive relations are at least valid for the dilute granular gases whose coefficient of restitution, $\tilde{e}$, is not too small. Moreover, the temperature dependence of the viscosity and the heat conductivity can be found [up to $O(1)$ coefficients], by

mean free path considerations (in the dilute limit), cf., e.g., Ref. [9]. The stress tensor, as proposed in Ref. [6] assumes the standard hydrodynamic form $\mathbf{P} = -p_h\mathbf{I} - 2\mu(\mathbf{D} - \frac{1}{3}\,\mathrm{div}\mathbf{I})$, where $\mathbf{I}$ is the unit tensor, p_h the pressure, and μ the viscosity. The heat flux $\mathbf{Q}$ is given by $\mathbf{Q} = \kappa\nabla T$. The pressure is given by $p_h = a\rho T$, the viscosity by $\mu = b\sigma\rho_s T^{1/2}$, the heat conductivity by $\kappa = c\sigma\rho_s T^{1/2}$ and $\gamma = (d\rho_s/\sigma)T^{3/2}$, where σ is the diameter of a particle and ρ_s is the density of a solid particle. The quantities a, b, c, and d are functions of the solid volume fraction alone. Their dilute limit values are $a = 1$, $b = 5\pi^{1/2}/48$, $c = 25\pi^{1/2}/128$, and $d = 24\epsilon v^2/\pi^{1/2}$, where $v = \rho/\rho_s$ is the volume fraction and [10] $\epsilon \equiv 1 - \tilde{e}^2$. The form of γ is easy to understand since the flux of particles impinging on a given particle is proportional to $T^{1/2}$ and the energy loss per collision is proportional to T. Consider now the following ("basic") homogeneous, statistically quiescent, solution of Eqs. (1)–(3): $\mathbf{v} = 0$, $\nabla T = \nabla\rho = 0$, and $T(t) = T(0)/(1 + t/t_0)^2$, where $t_0 = \pi^{1/2}\sigma/8v\epsilon T^{1/2}(0)$. This basic solution is unstable to linear perturbations. A linearization of Eqs. (1)–(3) around the basic solution yields equations with space-independent (but time-dependent) coefficients. In an infinite (or periodic) system the eigenmodes of the linearized equations are plane waves. Each Fourier component of the density field corresponds to a combination of eigenmodes of which one is unstable (hence the density Fourier modes are unstable), growing as $(1 + t/t_0)^{1 - k^2l^2/3\epsilon^2}$, for $kl \ll \epsilon$ and as $(1 + t/t_0)^{3/5 - 5k^2l^2/8\epsilon}$ for $\epsilon \ll kl < 1$, where k is a wave number and $l = \pi^{1/2}\sigma/8v$ is proportional to the mean free path. The vorticity, curl$\mathbf{v}$, is decoupled from the other modes and it decays as $(1 + t/t_0)^{-5k^2l^2/12\epsilon}$. The other hydrodynamic modes decay in a similar fashion to that of the vorticity. The decay rate of these is slower than that of the (homogeneous) temperature (the basic solution) for $kl < O(\epsilon^{1/2})$. Thus, following an initial transient the hydrodynamic modes dominate the dynamics of the system and the linear analysis around the basic solution is no longer valid.

Consider now the nonlinear stage of evolution in the following way. Notice that for times $t \ll t_0$ and $kl < \epsilon$ the linear modes (born as fluctuations) are practically stationary. Consider now a shear fluctuation corresponding to a typical wave number k and a typical velocity (amplitude) u such that its decay rate is much slower than that of the basic temperature field. Consider the nonlinear evolution of this perturbation. Assume that the temperature convection and conduction terms can be neglected in Eq. (2), an assumption to be justified a posteriori. Also, since div$\mathbf{v} = 0$ for the shear mode, the viscous heating term in this equation becomes $b\sigma\rho_s T^{1/2}h$, where $h \equiv \mathrm{tr}(\hat{\mathbf{D}})^2$ and $\hat{\mathbf{D}} \equiv \mathbf{D} - \frac{1}{3}\,\mathrm{tr}(\mathbf{D})\mathbf{I}$ is the deviatoric stress tensor. Consequently, one obtains from Eq. (2) and the values of l and ϵ defined above (for a dilute system)

$$\dot{T} = \frac{5}{9}lhT^{1/2} - \frac{2\epsilon}{l}T^{3/2}. \tag{4}$$

For a fixed value of the density ρ and of h, the solution of Eq. (4) is

$$T(t) = \frac{5l^2h}{18\epsilon}\left[\frac{1 - A\exp[-t(10/9)\epsilon h]}{1 + A\exp[-t(10/9)\epsilon h]}\right]^2, \tag{5}$$

where A is a constant depending on the initial condition $T(0)$. The initial time ($t = 0$) can be set to be the formation time of the shear fluctuation. Thus, the temperature saturates to the value (dictated by the viscous heating rate) $5l^2h/18\epsilon$ on a finite time scale of $t_1 = (\frac{10}{9}h\epsilon)^{-1/2}$, whereas on the linear level all quantities depend on time through powers of time. The above mentioned typical amplitude of the shear mode, u, can be defined through $h = k^2u^2$. During the time t_1, the (linear) shear mode decays by a factor

$$\left[1 + \left[\frac{9}{10}\right]^{1/2}\frac{T^{1/2}(0)}{u}\frac{\sqrt{\epsilon}}{kl}\right]^{-5k^2l^2/12\epsilon}.$$

This factor is close to unity provided $kl/\sqrt{\epsilon} \le u/\sqrt{T(0)}$. Since at the (initial) linear stage of evolution, the temperature decays faster than the hydrodynamic modes, this condition is easily met for wavelengths satisfying $kl \le \sqrt{\epsilon}$.

The condition that the shear mode is practically stationary until the temperature reaches the above mentioned saturation value (its linear equation being $\rho\,\partial\mathrm{curl}\mathbf{v}/\partial t = b\sigma\rho_s T^{1/2}\frac{1}{2}\Delta\mathrm{curl}\mathbf{v}$ with $T = 5l^2h/18\epsilon$) is that the typical time for the decay of the shear mode exceeds $(\frac{10}{9}k\epsilon)^{-1/2}$. This condition implies $k^2l^2 < (48/5\sqrt{2})\epsilon$. Defining the mean free path l as $l \approx 1/\sigma_T n$, where σ_T is the total cross section for collision and n is the number density, it follows that $l = \frac{3}{4}\sqrt{\pi}l$. Hence the above condition reads $k^2l^2 < (32/5\sqrt{2})\epsilon$.

The neglect of the heat conduction term in deriving Eq. (5) [with respect to $(d\rho_s/\sigma)T^{3/2}$] is justified provided $k^2l^2 < \frac{48}{25}\epsilon$. Similarly, the neglect of the thermal convection term is justified for $kl < 2\sqrt{T}(0)\epsilon/u$. Finally, the density can be considered to be constant during the time t_1 if $kl > T^{1/2}(0)\sqrt{\epsilon}/u$ or if $k^2l^2 \le \frac{24}{25}\epsilon$; the first of these two conditions is easily met, as explained above. All in all, the condition assuring the enslaving of the temperature to the shear field is $kl \le \sqrt{\epsilon}$. Once the temperature field is enslaved by the hydrodynamics (shear mode), the pressure balance is violated. Following the equation of state specified above the pressure corresponding to the saturation value of the temperature is $p_h = a\rho T \approx \rho T = \frac{5}{18}\rho l^2h/\epsilon = 5\pi\sigma^2\rho_s^2/18\times 64h/\epsilon\rho$, where $\rho = \rho_s v$ was used. Thus, the lower the pressure the higher the density. In the first stages of the density buildup, ρ is practically constant and the minima of the pressure correspond to the minima of the viscous heating function, h (i.e., the minimal shear). In a system in which the shear mode is characterized by a given wavelength, the heating function must thus have half this wavelength (being essentially the square of the vorticity), a fact which determines the typi-
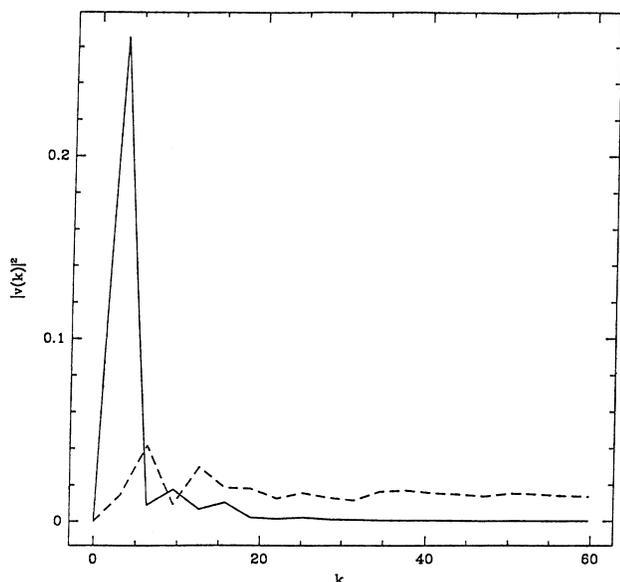
FIG. 1. Velocity power spectra for a rectangle of aspect ratio 2 (lengths 1 and 2 in dimensionless units). The coefficient of restitution is 0.92, the number of particles is 20000, and the area fraction of the disks is 0.05. The initial condition is homogeneous with a Maxwellian velocity distribution. The dashed line corresponds to five collisions per particle following the initial condition and the solid line, to 500 collisions per particle.

FIG. 2. Particle positions following 500 collisions per particle. Here $\bar{e} = 0.98$, and the number of particles is 20000.

cal scale of mass variation; this consequence of the analysis is corroborated by numerical simulations [8]. A full analysis of the time scale for mass motion towards the location of minimal pressure is rather lengthy. A simple way of obtaining this scale is by considering only the part of the equation of motion for the momentum density, $\mathbf{p} = \rho \mathbf{v}$, which is determined by the pressure, p_h, i.e., $\dot{\mathbf{p}} = -\nabla p_h$ in conjunction with the equation of continuity $\dot{\rho} = -\text{div}\,\mathbf{p}$. It follows that $\ddot{\rho} = -\text{div}\dot{\mathbf{p}} = \Delta p_h$. For a typical wave number k, the time scale for mass migration is thus $t_m = k^{-1}(\rho/p_h)^{1/2}$. Using $p_h \approx \frac{5}{18}\rho l^2 h/\epsilon$, $t_m = \sqrt{18/5}\epsilon^{1/2}/klh^{1/2}$. Notice that t_m is smaller, the larger k (it takes less time for mass to move a distance $2\pi/k$, the shorter the distance). On the other hand, cluster formation is possible only if $kl \leq \epsilon^{1/2}$. Hence the fastest, and consequently dominant, cluster formation process occurs at $kl \approx \epsilon^{1/2}$, i.e., the typical cluster separation distance is $O(l/\epsilon^{1/2})$. When $l/\epsilon^{1/2}$ is larger than the system size, one does not expect clustering to occur. Instead, one expects a shear mode and a density fluctuation mode of the smallest wave number allowed by the system's geometry.

The mechanism described above has been tested by simulating the (planar) dynamics of a collection of hard disks of uniform radius whose collisions are characterized by a fixed coefficient of normal restitution $\bar{e}$. The 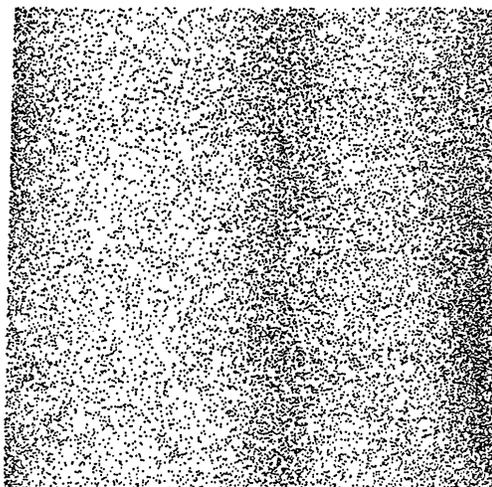geometry is rectangular and periodic boundary conditions are imposed. The system is *not subject* to any external forcing. The initial condition [11] is homogeneous in density and temperature and isotropic and homogeneous in the velocities. The initial velocity distribution is Maxwellian. The method of simulation is the "event-driven simulation," which is documented, e.g., in Ref. [12]. The simulations typically involve $N = 20000$ particles. We have computed a coarse-grained velocity field $\mathbf{v}$ by dividing the system into 32×32 cells and computing the ratio of the total momentum to the total mass in each cell. Typical results of several simulations are now described. Figure 1
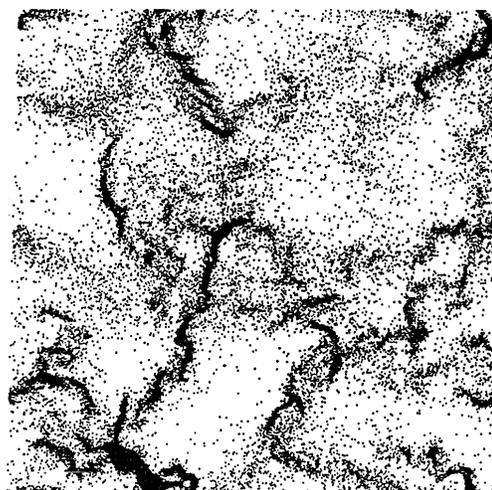

FIG. 3. A typical configuration of particles exhibiting clusters. Here the coefficient of restitution is 0.6, the time corresponds to 500 collisions per particle, and the area fraction is 0.05. The number of particles is 40000.

shows the velocity spectrum $|v(\hat{k})|^2$ (averaged over the directions of k) as a function of k. At early stages equipartition of energy is prominent, whereas at a later time a typical scale corresponding to $k = \pi$ in our units is observed (corresponding to dominance by the slowest shear model in the system). Figure 2 presents a typical mass distribution of the disks after a long enough time, for a system whose size is less than $l/(1 - \bar{e}^2)^{1/2}$, i.e., less than l/ϵ. It exhibits a density nonuniformity but no discernible clusters. Figure 3 corresponds to a system size larger than l/ϵ and the clusters are very prominent. Our results may be relevant to rapid granular flows of engineering importance since the clusters can affect the stresses and flow rates in various devices which transport solids. It is possible that the mechanism described here also bears some relevance to the process of planet formation, in particular to the process of coagulation to planetesimal formation.

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[2] For astrophysical applications, cf. *The Formation and Evolution of Planetary Systems*, Proceedings of the Formation and Evolution of Planetary Systems Meeting, Baltimore, 1988, edited by H. A. Weaver and L. Danly (Cambridge Univ. Press, Cambridge, 1989).

[3] Clusters in granular flows were observed numerically by M. A. Hopkins and Michel Y. Louge, Phys. Fluids A **3**, 4 (1990); I. Goldhirsch, in *Proceedings of the Joint DOE/NSF Workshop on Flow of Particulates and Fluids, Worcester, Massachusetts, 22–24 October 1991*, edited by S. I. Plasynski, W. C. Peters, and M. C. Roco (National Technical Information Service, Springfield, VA, 1991); O. R. Walton, H. Kim, and A. D. Rosato, in *Mechanics Computing in 1990's and Beyond*, edited by H. Adeli and R. L. Sierakowski (ASCE, New York, 1991), Vol. 2, p. 1249, and in a recent work on one-dimensional inelastic gases by S. McNamara and W. R. Young, Phys. Fluids A **4**, 496 (1992).

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[5] T. M. Mello, P. H. Diamond, and H. Levine, Phys. Fluids A **3**, 2067 (1991).

[6] J. T. Jenkins and M. W. Richman, Arch. Rat. Mech. **87**, 355 (1985); see also Ref. [7].

[7] J. T. Jenkins and S. B. Savage, J. Fluid Mech. **130**, 187 (1983).

[8] I. Goldhirsch (unpublished).

[9] M. Babic and H. Shen, J. Eng. Mech. **115**, 1262 (1989).

[10] In the original equations ϵ is taken to be $1 - \bar{e}$. Since the energy dissipated in a single collision is proportional to $1 - \bar{e}^2$, on the average, we prefer to choose $\epsilon = 1 - \bar{e}^2$.

[11] The initial condition is prepared in two steps. First a random generator is used to assign the particles random positions (in the rectangle) and random velocities whose distribution is Maxwellian. The next step consists of performing a run in which the collisions are taken to be elastic for about 100 collisions per particle to ensure an initial equilibrium state (uniformity of density is then tested by computing the time average of the density in each of the 32×32 cells, cf. text; the isotropy and Maxwellian nature of the initial condition was tested in a similar way as well).

[12] D. C. Rapaport, J. Comput. Phys. **34**, 184 (1980), and references therein.