Nonperturbative Solutions of String Theory in Gravitational Backgrounds

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We study bosonic string theory in a gravitational background. We show that either left-movers or right-movers are the only background independent nonperturbative solutions of the field equations for an arbitrary static metric. They are stable and have a conserved topological charge, therefore being topological solitons. The action vanishes for these solutions and hence they provide the dominating contribution in a path integral quantization.

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The study of nonperturbative solutions in classical field theories has shed light on many features of the corresponding quantum theory since in general these features cannot be reached by perturbation theory. For instance, topological solitons indicate the existence of hidden sectors of the Hilbert space while the presence of instantons indicates vacuum tunneling, showing, in both cases that a rich quantum structure is available. Since there is no manageable field theory for strings the only way to study the interaction of strings is through perturbation theory. Therefore any nonperturbative solution of string theory may give some clues about the symmetries or the geometrical foundations of a string field theory.

Another aim of studying strings in backgrounds consists in finding a (low energy) phenomenologically acceptable string vacuum or consistent compactifications. However, as is well known, strings do not propagate in arbitrary backgrounds if conformal symmetry is to be kept at the quantum level [1].

Since we find background-independent solutions conformal invariance can be achieved by choosing an appropriate background. Plane wave backgrounds [2], of which shock wave backgrounds [3] are a particular case, as well as group manifolds [4], are examples of backgrounds which keep conformal invariance at the quantum level. Recently, a class of space-time metrics has been proved finite in the presence of the dilaton field [5].

In this paper we study nonperturbative solutions of the bosonic string theory in a gravitational background field. We show that closed string left-movers or right-movers are background-independent solutions. However, not all coordinates can be taken as right- or left-movers since in this case the gauge freedom cannot be fixed completely. In the light-cone gauge (LCG) we find that only left- or only right-movers are solutions for static, but otherwise arbitrary backgrounds, and that these backgrounds can be interpreted as a comoving frame for the string. When both movers are present the space-time metric splits up in blocks with two of them still being completely arbitrary.

In fact, all coordinates being either left- or right-movers corresponds to the *only* background-independent solution. This solution is stable under small perturbations. Finally we show that the solutions having only the string winding mode can be interpreted as solitons and a quantum soliton operator can be built. Since the action vanishes for left- or right-movers they give the dominating contribution to the path integral.

The string action in an arbitrary D-dimensional gravitational background $G_{\mu\nu}$ ($\mu, \nu = 0, \ldots, D - 1$) is given by

$$S = \frac{1}{2\pi\alpha'} \int d\sigma \, d\tau \sqrt{g} \; g^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu, \quad (1)$$

where $g_{\alpha\beta}$ ($\alpha, \beta = 0, 1$) is the world-sheet metric. The classical equations of motion and constraints are, respectively,

$$\Box^2 X^\mu + \Gamma^\mu_{\nu\lambda} g^{\alpha\beta} \partial_\alpha X^\nu \partial_\beta X^\lambda = 0, \quad (2)$$

$$T_{\alpha\beta} = G_{\mu\nu}(\partial_\alpha X^\mu \partial_\beta X^\nu - \tfrac{1}{2} g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu)$$
$$= 0, \quad (3)$$

where $\Gamma^\lambda_{\mu\nu}$ is the Christoffel symbol for the metric $G_{\mu\nu}$. In the conformal gauge, (2) and (3) reduce to

$$\hat\partial_+ \hat\partial_- X^\mu + \Gamma^\mu_{\nu\lambda} \hat\partial_+ X^\nu \hat\partial_- X^\lambda = 0 , \quad (4)$$

$$T_{\pm\pm} = G_{\mu\nu} \hat\partial_\pm X^\mu \hat\partial_\pm X^\nu = 0 , \quad (5)$$

where $\hat\partial_\pm \equiv \partial_\tau \pm \partial_\sigma$.

As is easily seen, right- or left-movers, satisfying $\hat\partial_+ X^\mu = 0$ or $\hat\partial_- X^\mu = 0$, respectively, are background-independent solutions of the field equations (4). The Euclidean version of these solutions has been recognized as instantons in two-dimensional nonlinear sigma models for target spaces with nontrivial second homotopy group [6]. In our case, however, there are two more equations to be satisfied, namely, the constraints (5). As will be shown, they prove to be very strong, restricting the allowed back-

ground space-times.

In two-dimensional Minkowsky space-time, fields satisfying either of the conditions $\hat{\partial}_+ X^\mu = 0$ or $\hat{\partial}_- X^\mu = 0$ are known as chiral bosons. They have been studied and their quantization performed in several schemes [7]. In the heterotic string theory, left-mover target space coordinates generate the internal symmetry group through compactification on a flat torus [8], whereas in the context of nonlinear sigma models, chiral bosonic fields—the so-called leftons and rightons—have also been considered [9]. We shall denote string coordinates of only a mover type (either left or right) as being "chiral."

As is well known the gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ still leaves a combination of reparametrization invariance and Weyl scaling to be fixed. This freedom enables us to make a further gauge choice and the light-cone gauge $X^+ \equiv \frac{1}{\sqrt{2}}(X^0 + X^{D-1}) = x_0^+ + P^+\tau$ is taken in order to fix

this residual invariance [10]. In this gauge, however, the coordinate X^+ is no longer chiral. We could therefore ask whether it is possible to obtain a solution with all coordinates being chiral with a gauge choice such as $X^+ = x_0^+ + P^+\sigma^+$ (where $\sigma^\pm = \sigma \pm \tau$). Nevertheless, this gauge does not completely fix the residual gauge freedom left over by the conformal gauge, namely, $\sigma^+ \to \tilde{\sigma}^+(\sigma^+)$, $\sigma^- \to \tilde{\sigma}^-(\sigma^-)$, as can be easily checked.

We now turn to study the backgrounds where not all coordinates are chiral. In order to fix the residual gauge invariance we choose the LCG with the nonchiral coordinate X^+. The LCG choice will affect the equations of motion (4) where all terms containing X^+ will spoil the possibility of a chiral solution for the remaining coordinates. Starting with a metric such that $G_{++} = G_{--} = 0$, which partially sets the coordinate system [11], we can require that these various terms containing X^+ vanish. For right-movers this leads to the equations

$$2\,P^+\partial_+ G_{+-} + (\partial_+ G_{-i} + \partial_i G_{+-} - \partial_- G_{+i})\hat{\partial}_- X^i = 0, \tag{6a}$$

$$2\,P^+\partial_+ G_{+i} + (\partial_+ G_{-i} + \partial_- G_{+i} - \partial_i G_{+-})\hat{\partial}_- X^- + (\partial_+ G_{ij} + \partial_{[j} G_{+i]})\hat{\partial}_- X^j = 0, \tag{6b}$$

where $\partial_\pm$ refers to space-time derivatives (while $\hat{\partial}_\pm$ refers to world-sheet derivatives). The constraints (5) then imply

$$2P^+ G_{+-}\hat{\partial}_- X^- + 2P^+ G_{+i}\hat{\partial}_- X^i + 2G_{-i}\hat{\partial}_- X^-\hat{\partial}_- X^i + G_{ij}\hat{\partial}_- X^i\hat{\partial}_- X^j = 0. \tag{6c}$$

Now we must solve (6) for the various components of the metric as functions of X^+, X^-, and X^i. It turns out that these equations have a solution for a static (i.e., X^+ independent) background

$$\begin{aligned} G_{ij} &= g_{ij}, \quad G_{+i} = \partial_i h, \\ G_{+-} &= \partial_- h, \quad G_{-i} = g_{-i}, \end{aligned} \tag{7}$$

where g_{ij}, g_{-i}, and h are functions of X^- and X^i only. Notice that, apart from being static, (7) corresponds to a choice of a coordinate frame, as the following counting of its independent components shows. The spatial metric g_{ij} has $\frac{1}{2}(D-2)(D-1)$ independent components while g_{-i} has $D-2$ and h, one. Altogether they give the $\frac{1}{2}D(D-1)$ independent components of a D-dimensional metric after a choice of a coordinate frame is made.

The static character of the solution (7) stems from the fact that the chiral coordinates X^- and X^i depend symmetrically on the world-sheet parameters σ and τ, while in the LCG this symmetry is absent in X^+. Therefore, an equation of motion for the string coordinates cannot in principle be satisfied once the background depends on X^+, but must be static. It can also be understood with the following argument. If we start with a flat space-time where there is a closed string with both left- and right-movers, then we could go over to a curved background where only one of the movers remains, according to our solution. As this cannot happen, it means that it is not possible to curve the flat space. Therefore we must have a static background, or at least a background with a time dependence that can never reach flat space. If we insist

on curving the background we would find a solution with both movers mixed up nonlinearly in order to satisfy the equations of motion (4) and the constraints (5).

An important feature of our solution is that the center-of-mass momentum is always equal (up to a sign) to its winding number, as can be seen, for instance, from the right-mover

$$\begin{aligned} X_R^i(\sigma, \tau) &= x_R^i + \left(\frac{P^i}{2} - L^i\right)(\tau - \sigma) \\ &\quad + \frac{i}{2}\sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}_n e^{-2in(\tau-\sigma)}, \end{aligned} \tag{8}$$

where x_R^i is the zero mode and P^i and L^i are the momenta and winding modes of the associated nonchiral string which now appear combined to give what is both the momenta and the winding number of the chiral string. This makes the string slip over itself in such a way that its center of mass is always at rest. In fact, the class of metrics (7) corresponds to a comoving frame for the string.

For a relativistic point particle the comoving frame is defined by [12]

$$ds^2 = dt^2 - G_{ij}(X, t)dX^i dX^j. \tag{9}$$

Once we go to the proper-time gauge $X^0 = P^0\tau$ (it plays a role similar to the conformal and the LCG for the string), the equations of motion for the point particle take the form

$$\ddot{X}^i + G^{ik}\dot{G}_{kj}\dot{X}^j - \tfrac{1}{2}G^{ij}\partial_j G_{kl}\dot{X}^k\dot{X}^l = 0 , \qquad (10a)$$

and the point particle constraint, $P^2 = m^2$, becomes

$$(P^0)^2 - G_{ij}\dot{X}^i\dot{X}^j = m^2, \qquad (10b)$$

where m is the particle rest mass. The only background-independent solution to these equations is $\dot{X}^i = 0$. To see this take the partial derivative of the constraint (10b),

$$(\partial_k G_{ij})\dot{X}^i\dot{X}^j = 0. \qquad (11)$$

Thus, (10a) gives $\partial_\tau(G_{ij}\dot{X}^j) = 0$, or $\dot{X}^j = G^{jk}C_k$, for a constant C_k. Substituting in (10b) gives $C_k C_l G^{kl} = cte$. Since G_{kl} is arbitrary, this identity can only be satisfied if $C_k = 0$, or $\dot{X}^i = 0$. The equation of motion for X^0 (namely, $\dot{X}^i\dot{X}^j\partial_0 G_{ij} = 0$) is consistent with (10b) and was not directly used in this derivation.

For the string, it can be easily demonstrated, with a metric of the form

$$G_{\mu\nu}(X^i) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & G_{ij} \end{pmatrix}, \qquad (12)$$

that either left- or right-movers are the unique background-independent solutions.

The equation of motion for X^+ gives $\partial_- G_{ij} = 0$, while the equation of motion for X^- (which is the same as for the flat case), for the X^i, and for the constraints are, respectively,

$$\hat{\partial}_+\hat{\partial}_- X^- = 0, \qquad (13a)$$

$$\hat{\partial}_+\hat{\partial}_- X^i + \tfrac{1}{2}G^{il}(\partial_j G_{kl} + \partial_k G_{jl} - \partial_l G_{jk})\hat{\partial}_+ X^j\hat{\partial}_- X^k = 0, \qquad (13b)$$

$$2P_+\hat{\partial}_\pm X^- = G_{ij}\hat{\partial}_\pm X^i\hat{\partial}_\pm X^j. \qquad (13c)$$

From (13a) and (13c) one has

$$G_{ij}\hat{\partial}_\pm X^i\hat{\partial}_\pm X^j = F(\sigma^\pm). \qquad (14)$$

This last equation allows us in fact to have both movers. To take this into account let us split the coordinates X^i into left-mover coordinates X^A ($\hat{\partial}_- X^A = 0, A = 1, ..., D' - 1$) and right-movers X^a ($\hat{\partial}_+ X^a = 0, a = D', ..., D - 2$). Without loss of generality, the components of the metric G_{ij} can also be split into G_{ab}, G_{AB}, and G_{aB}. Then from (14) we obtain G_{AB} as an arbitrary function of X^A and similarly for G_{ab}, while from (13b) one gets

$$(\partial_B G_{Ab} - \partial_A G_{Bb})\hat{\partial}_- X^b\hat{\partial}_+ X^B = 0 \quad \text{for } i = A,$$
$$(\partial_b G_{aB} - \partial_a G_{bB})\hat{\partial}_- X^b\hat{\partial}_+ X^B = 0 \quad \text{for } i = a . \qquad (15)$$

A local solution for the mixed part of the metric is $G_{aB} = \partial_a\partial_B f$ where f is an arbitrary function of X^A and X^a. The background allowing both movers is then

$$G_{AB} = G_{AB}(X^A), \quad G_{ab} = G_{ab}(X^a) ,$$
$$G_{aB} = \partial_a\partial_B f(X^A, X^a). \qquad (16)$$

By substituting (16) in the equations of motion one does indeed see that this is the most general class of

metric which allows the presence of both movers. Nevertheless, for an *arbitrary* metric G_{ij}, *only* the solution with one of the movers present is allowed. For metrics of the form (7) the chiral solution is *also* unique. This stems from the fact that an arbitrary background of type (7) is more general than an arbitrary background of type (12).

Now we turn to the study of the stability of the solution. Since our solution can be regarded as, say, a right-mover closed string with its center of mass at rest in a comoving frame we have to look for small perturbations which can grow up with time τ or small perturbations which can change the chirality of the string. The question of stability can then be posed as a problem of finding the eigenvalues w for a perturbation $f(\sigma)$ with small coefficient λ,

$$X^i(\sigma, \tau) = X^i(\sigma - \tau) + \lambda f^i_w(\sigma)e^{-iw\tau}. \qquad (17)$$

As usual, real eigenvalues indicate the stability of the solution. By substituting it into the equations of motion and keeping only the linear terms it can then be easily seen that, due to the dependence on τ of the Christoffel symbol (and coordinate X^j),

$$f''^i(\sigma) + w^2 f^i(\sigma) = -2[\Gamma^i_{jk}\dot{X}^j](\sigma, \tau)[iw f^k(\sigma) - f'^k(\sigma)], \qquad (18)$$

the eigenvalue equations decouple and we get $f(\sigma) = e^{iw\sigma}$ with w real. Then the perturbation does not change the character of the solution (it remains a right-mover) and we conclude that the solution is stable under small perturbations.

Besides having zero action, which means a large contribution to the path integral, another important feature of the chiral solution in curved space is its solitonic character. The modes of the string are not affected by the presence of the background, a behavior that resembles solitons. The same occurs with the winding mode, which, in addition, has a topological character. On the other hand, the Hamiltonian

$$H = \int d\sigma \ G_{ij}\dot{X}^i\dot{X}^j = \int d\sigma \ G_{ij}X'^i X'^j \qquad (19)$$

does depend on the metric. This can be interpreted even if the string is understood as being in a comoving frame, since it is an extended object and its energy is not localized.

As in flat space-time [13] the string winding mode has a solitonic character. Since we have a topological conserved current $J^i_\alpha = (2\pi)\epsilon_{\alpha\beta}\partial^\beta X^i$, there is a conserved charge for a solitonic solution of the form $X^i_{\text{sol}}(\sigma - \tau) = 2w^i(\sigma - \tau)$, where w^i is the winding number. A formal expression for the quantum soliton operator which creates a soliton (of winding number w^i) out of the vacuum is given by

$$T(w) = \exp\left(-\frac{i}{\pi}\int d\sigma^- \ X^i_{\text{sol}}(\sigma - \tau)P_i(\sigma - \tau)\right) , \qquad (20)$$

where P_i is the momentum associated to X^i, i.e., $P_i = G_{ij}\hat{\partial}_- X^j$. This operator satisfies the usual commutation rule with the charge operator $\hat{W}^i = (2\pi)^{-1} \int d\sigma^- \; \hat{\partial}_- X^i$, that is,

$$[\hat{W}^i, T(w)] = w^i T(w). \tag{21}$$

We conclude with some comments and remarks. As far as the properties of the string solution are concerned only local properties of the space-time, like its curvature, are relevant. Topological properties such as a possible nonsimply connected structure or compactedness of the background are not relevant to select one of the movers. The later properties may contribute to a nontrivial fundamental homotopy group which in turn determines the presence of winding modes only. In connection with this, other backgrounds such as the antisymmetric tensor can be considered. It will contribute a torsion term to the equations of motion.

The solution allowing both movers Eq. (16) could be turned into a Kähler metric if $G_{AB} = G_{ab} = 0$ and if a complex structure is properly introduced after Euclideanization. The left- or right-movers turn into holomorphic or antiholomorphic functions and one may interpret them as sigma-model instantons defined on a Kähler manifold [14]. However, this interpretation is not valid since our base space is compact (in the world-sheet coordinate σ) and it is not possible, by joining a point at infinity, to define the second homotopy group needed for the characterization of instantons.

Besides the static backgrounds which allow left- and/or right-movers, we also found from Eqs. (6a), (6b), and (6c) a time-dependent solution

$$G_{+-} = \partial_- h \,, \tag{22a}$$

$$G_{-i} = [a_i X^+ + g_i(X^i)]G_{+-} \,, \tag{22b}$$

$$G_{+i} = a_i h + \partial_i h \,, \tag{22c}$$

$$G_{ij} = \frac{G_{+(i} G_{-j)}}{G_{+-}} \,, \tag{22d}$$

where the a_i are constants, $g_i = g_i(X^i)$, and $h = h(X^-, X^i)$. As pointed out above such a time-dependent solution should not be allowed. In fact, it is possible to show that the metric (22) is singular. Singular metrics are allowed in gauge theories for gravitation but their role is not understood [15]. Moreover it should be noticed that such a metric admits black hole solutions since, if G_{+-} vanishes in some point, then G_{ij} becomes singular

(and vice versa).

Other background fields may be considered. The antisymmetric tensor, for instance, contributes to the equations of motion with a torsion which is the antisymmetric part of the Christoffel symbol. This generalized equation still allows the chiral solution.

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