Smallest Universe of Negative Curvature

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The smallest known three-dimensional closed manifold of curvature $k \equiv -1$ was discovered a few years ago by Weeks. This kind of manifold is constructed from a hyperbolic polyhedron with faces pairwise identified. Here it is used as the comoving spatial section of a Friedmann cosmological model, in the spirit of Ellis and Schreiber's idea of small universes. Its nontrivial global topology has the effect of producing multiple images of single cosmic sources, and this is the basis of an attempt to solve a famous controversy about the redshifts of quasars.

The idea that cosmic space may have a nontrivial global topology, being *closed* (i.e., finite and boundless) for any sign of the spatial curvature, has had a slow but sustained development since Ellis's pioneer paper [1]. More recent literature can be traced from Fagundes [2]. The usual assumptions of theoretical cosmology, homogeneity and isotropy of matter distribution (cf. Landau and Lifshitz [3], for example), combined with general relativity, only determine the local spacetime metric, but not the global topology of spacetime. For subcritical densities ($\Omega < 1$) and null cosmological constant Λ, the Friedmann-Robertson-Walker metric is $ds^2 = -c^2 dt^2 + a^2(t) d\lambda^2$, where

$$d\lambda^2 = d\chi^2 + \sinh^2\chi (d\chi^2 + \sin^2\theta \, d\varphi^2) \tag{1}$$

is the metric of comoving space, with constant curvature $k = -1$. The usual comoving spatial section attributed to this solution is the open, simply connected, infinite hyperbolic space H^3, but there is an infinity of closed, multiply connected hyperbolic manifolds [4] (CHMs) that can be taken as that spatial section. A CHM is usually constructed [1] from a *fundamental polyhedron* (FP) or *Dirichlet region* in H^3, with faces pairwise identified. The FP determines a *tessellation* or *honeycombing* [5] of H^3 into cells which are replicas of the FP, through the action of a discrete group of isometries related to the face pairings (see below) and to the manifold's topology.

Some time ago the author (Fagundes [6]; henceforth that paper will be referred to here as QGAII, after its title) tried to solve the quasar redshift controversy [7] through a model with nontrivial topology of three-space: The multiple images of single sources predicted by these universes might explain the presumed associations of quasars and galaxies with discordant redshifts. That model's CHM had a hyperbolic icosahedron as FP. The results were qualitatively interesting, but quite off the observed redshifts: In hyperbolic geometry one has no freedom to fix the scale of such polyhedra, because their dihedral angles are fixed by the requirements of the mentioned tessellation; and the big size of that icosahedron (volume $V = 4.686\ldots$ in the usual normalization of the curvature to $k = -1$) did not allow the production of Arp's pairs

with realistic redshifts.

In this paper I build a similar model [8] with the smallest CHM that is known to geometers at present, $V = 0.9427\ldots$ This manifold was discovered by Weeks [9], so it will be referred to as *the Weeks manifold* (WM). Its small size is particularly interesting to cosmology, in the light of Ellis and Schreiber's [10] proposal of a *small universe* and of the above remark about the redshifts of quasars associated with galaxies. My choice of a subcritical density, $\Omega = 0.1$, is mainly dictated by the stated concern of finding a theory for these and similar associations as a topological effect, but also by a desire to explore techniques for dealing with the geometry of CHMs. Note that a low value for Ω is consistent with astrophysical observations and inferences, although theoreticians have been giving preference to the critical value $\Omega = 1$, based on speculative forms of dark matter.

The FP for the Weeks manifold is an 18-sided figure whose 26 vertices' *Minkowski coordinates* in my representations are listed in Table I. These coordinates are related to the spherical ones in Eq. (1) by $X_1 = \sinh\chi \sin\theta \times \cos\phi$, $X_2 = \sinh\chi \sin\theta \sin\phi$, $X_3 = \sinh\chi \cos\theta$, $X_4 = \cosh\chi$, point $\chi = 0$ being the centroid of the FP. They are very useful for handling rigid motions in H^3, for this space is imbedded in an abstract Minkowski space as the upper branch of a hyperboloid [3]. So a rigid motion in H^3 is expressed as a proper, isochronous Lorentz transformation [11]. The group Γ which produces the tessellation for WM is generated by 18 such transformations γ_k, $k = 0$-17. Each of these is a motion that takes the FP to a replica γ_k (FP), which has a face in common with the original; the faces are so numbered that for $k = 0$-8, $\gamma_k(F_{k+9}) = \gamma_k(\text{FP}) \cap \text{FP} = F_k$ and $\gamma_{k+9} = \gamma_k^{-1}$. Table II lists the faces of FP, and the generators that link them, expressed in terms of two more basic motions [12], a and b, given in Table III.

Ellis and Schreiber [10] mention several advantages of the cosmic three-space being closed and small. Among these is the fact that it helps explain the isotropy and homogeneity of cosmic observations, with or without the inflationary scenario. One may add that the production of multiple images, besides serving our current purpose

TABLE I. The Minkowski coordinates for the 26 vertices of the fundamental polyhedron for the Weeks manifold, which is the comoving spatial section of this paper's model.

	X_1	X_2	X_3	X_4
G	0.00000000	0.00000000	0.81159257	1.28789848
O	0.00000000	0.10664201	0.78401962	1.27517029
A	0.47654246	0.27513192	0.61537675	1.29671849
B	0.62025991	−0.16851487	0.16978597	1.20081093
D	0.43218382	0.62623778	0.21703147	1.27517029
T	0.68871325	0.03742749	−0.38772738	1.27517029
C	0.54663073	0.54283761	−0.29664965	1.29671849
X	0.65134973	−0.00044773	−0.48417540	1.28789848
W	0.57995255	0.06697606	−0.53406790	1.27517029
R	−0.09235469	−0.05332100	0.78401962	1.27517029
H	−0.47654246	0.27513192	0.61537675	1.29671849
L	−0.16419180	0.62141827	0.16978597	1.20081093
I	−0.75842974	0.06116328	0.21703147	1.27517029
Q	−0.37676978	0.57772942	−0.38772738	1.27517029
K	−0.74342652	0.20197730	−0.29664965	1.29671849
Y	−0.32528712	0.56430928	−0.48417540	1.28789848
U	−0.34797925	0.46876561	−0.53406790	1.27517029
S	0.09235469	−0.05332100	0.78401962	1.27517029
E	0.00000000	−0.55026383	0.61537675	1.29671849
J	−0.45606811	−0.45290341	0.16978597	1.20081093
F	0.32624592	−0.68740106	0.21703147	1.27517029
P	−0.31194347	−0.61515691	−0.38772738	1.27517029
M	0.19679579	−0.74481490	−0.29664965	1.29671849
Z	−0.32606261	−0.56386155	−0.48417540	1.28789848
V	−0.23197331	−0.53574168	−0.53406790	1.27517029
N	0.00000000	0.00000000	−0.66479086	1.20081093

TABLE II. The faces of the fundamental polyhedron and the generators that relate them: $F_k = \gamma_k(F_{k+9})$, $k = 0$–8. a and b are defined in Table III.

k	F_k	F_{k+9}	γ_k
0	ABTCD	HLDAO	$a^2 b^{-1}$
1	SEFBA	FMPJE	ab
2	RHIJE	IKQLH	$b^{-1}ab^{-1}$
3	FMXTB	UKZVN	a
4	WNUYC	IJPZK	b
5	CYQLD	MXWNV	ba^{-1}
6	CTXW	HOGR	$b^2 a^{-1}b$
7	KQYU	ERGS	$ba^{-1}b^2 a^{-1}b$
8	MPZV	ASGO	$b^{-1}a^{-1}b^{-1}a^{-1}$

position), the others being at points P_n, $n = 1$–30, chosen pseudorandomly. With each generator applied to these points we get $\gamma_k(P_n)$, $k = 0$–17, $n = 0$–30, as potential [16] indirect images; the figure represents all images with redshift $Z \leq 3.1$, as seen from P_0. The apparent anisotropy in the figure with some voids and concentrations is suggestive but is not meaningful in the present context, because we only have a partial covering of the space of images H^3, namely the FP itself and the 18 cells γ_k(FP). Now, a degree of anisotropy is expected in such cosmologies, as explained in QGAII and in agreement with Arp's observations [17]. Kokubun and I [18] are working on a more detailed computer simulation of the QGAII model, to estimate this anisotropy in the case of quasar-galaxy associations. In a future study a complete covering of the observable cosmos will be made using the present theory, to try to explain topologically the small anisotropy detected by NASA's COBE satellite in the 2.7 K background radiation [19].

As in QGAII, to interpret Arp's associations as conjunctions of pairs of images of single sources, we need to determine self-crossing geodesics in the comoving space. To find these, one scans each face of the FP, looking for points that, linked to Earth's position, produce self-crossing geodesics in the FP. The method is given in detail in Appendix B of QGAII, except that here we first triangulate the faces, which are either pentagonal or quadrilateral—cf. Table II. Earth's position was taken to be $\chi = 0.4$, $\theta = 15°$, $\varphi = 30°$. As in Fig. 1, solutions with

about the redshift controversy, is quite interesting by itself, for it would give us information about different epochs of the sources' evolution. And that, according to quantum cosmologists, the probability for *spontaneous creation* of a small universe is bigger than for a large one, being null for infinite universes [13].

Figure 1 is similar to but more detailed than Fig. 3 in Ref. [10], in that it represents the simulated distribution [14] of images produced by 31 sources in WM as cosmic space. In these models it is the distribution of *images*, not sources, that is expected to be approximately homogeneous and isotropic [15]. One of the sources is the Milky Way galaxy at point P_0 (the Earth's, or observer's,

TABLE III. The basic matrices in terms of which generators γ_k are expressed in Table II.

$$a = \begin{pmatrix} 1.08002794 & -0.23611344 & 0.89642406 & 1.01281091 \\ 0.01223938 & -0.72016273 & -0.96925233 & -0.67693001 \\ 0.47017146 & 0.65634321 & -0.62440977 & -0.20429190 \\ 0.62263260 & 0.07190498 & 1.06438418 & 1.58926252 \end{pmatrix}$$

$$b = \begin{pmatrix} -0.17317475 & 0.95965035 & 0.33332416 & 0.24904476 \\ 0.71129753 & 0.53303996 & -0.73535204 & 0.57516814 \\ -1.28760915 & -0.29169981 & -0.62440977 & -1.06438418 \\ 1.09264404 & 0.53865497 & 0.20429190 & 1.58926252 \end{pmatrix}$$
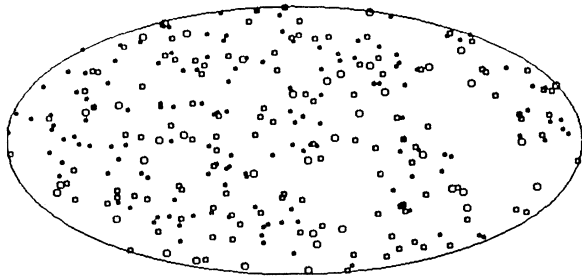
CONVENTIONS: 0 < Z < 1 ○   1 < Z < 2 ◉   2 < Z < 3.1 ●

FIG. 1. Celestial map of images produced by our Galaxy and thirty other "galaxies," in a simulation according to the model. Earth's position in the fundamental polyhedron is $\chi = 0.40$, $\theta = 15°$, $\varphi = 30°$.

$Z > 3.1$ were excluded.

The results [20] are presented in Table IV as exact conjunctions of images, but of course observed separations of up to a few arcminutes should be regarded as good data for comparison. Note that the rays producing these second images travel long paths of the same length as the distances of the corresponding images, and are sometimes scattered, sometimes focused by intervening gravitational fields. Each scanned face or *search region* produces a continuous line segment, whose points are the *potential* positions of objects in conjunction with one of its topologically generated images; the positions in the table are sample points along these segments—cf. Fig. 5 in QGAII.

A fit to the quasar-galaxy pairs in Burbidge *et al.* [21], for example, would be quite unreliable at this point, since besides arbitrary parameters in the model (Earth's position in the FP, the orientation of the axes, and the values of Λ and Ω), only a fraction of the observed pairs can be attributed to a topological effect, the majority being line-of-sight coincidences. One would like to see a deeper study of the observed pairs that might reveal evolutionary traits in a number of them. Then we would get a separation of accidental versus topological associations, and a meaningful fit of the model could become feasible.

However, the redshifts listed for the associations in Table IV are quite realistic, and the fact that the conjunctions occur in aligned groups also reflects some of Arp's claims, at least qualitatively. For example, the images produced by search region 10 have redshifts in ranges similar to those of the grouping of galaxies, quasars, and radio sources near galaxy M33. It is true that the group of region 10 covers a linear range of about 22°, with two classes of smoothly varying redshifts; while in Arp [7] the group is quite scattered about a line, and has a range of 50°. On the other hand, there are tight groups, like the three quasistellar objects ($Z = 2.048$, 2.054, 2.040) about the irregular galaxy M82 ($Z = 0.001$) [22], for which I gave a tentative interpretation

TABLE IV. Sample values of potential positions for two images of single sources in conjunction: Z_1, Z_2 are their redshifts; lon, lat their theoretical longitude and latitude.

Search region	Z_1	Z_2	lon	lat
0	0.800	2.591	15.56	−42.82
	0.516	2.041	19.28	−41.18
	0.337	1.723	22.99	−39.11
	0.215	1.531	26.42	−36.67
	0.123	1.409	29.51	−33.86
	0.046	1.335	32.22	−30.67
	0.002	1.306	33.75	−28.36
1	0.001	1.509	346.24	28.45
	0.048	1.519	347.53	30.77
	0.119	1.560	349.72	33.84
	0.205	1.643	352.46	36.81
	0.308	1.777	355.50	39.41
2	0.008	2.065	230.27	12.20
	0.094	2.017	231.24	15.72
	0.231	2.039	233.34	20.69
	0.373	2.158	235.75	24.75
	0.536	2.370	238.28	28.10
9	0.002	1.306	127.02	19.21
	0.021	1.317	127.41	20.33
	0.095	1.379	129.06	24.09
	0.189	1.494	131.24	27.76
	0.246	1.577	132.48	29.46
	0.356	1.755	134.56	31.92
10	1.001	3.046	251.20	−35.66
	0.761	2.569	252.81	−34.09
	0.526	2.131	255.14	−31.50
	0.393	1.906	256.91	−29.18
	0.239	1.683	259.41	−25.17
	0.050	1.520	262.59	−17.29
	0.000	1.509	263.22	−14.57
11	0.949	3.085	149.40	−36.25
	0.718	2.664	148.91	−34.29
	0.482	2.294	152.42	−30.97
	0.199	2.025	157.80	−23.75
	0.021	2.054	160.69	−17.01
	0.002	2.070	160.91	−16.23
	0.001	2.071	160.92	−16.19
	0.001	2.071	160.93	−16.18

in QGAII in terms of a collision of quasar-centered galaxies to form young M82 (now invisible); if we look at search region 11 in Table IV we do see some numbers like these. The separation between M33 and M82 is not reproduced in this way, but the tendency remains for topologically nontrivial cosmologies to account for these and other clusterings with discordant redshifts.

Some of the unorthodox claims this theory did not confirm: material bridges or jets linking objects with discordant redshifts [7]—but see Fagundes [23]; groups with small, discordant redshifts, like Stephan's quintet of gal-

axies, presumed to be physically associated—but Hickson, Kindl, and Huchra [24] have argued that there is a realistic chance that this could be just a coincidence of projections on the sky. Hayward and Twamley [25] use a lower bound for the volume of CHMs, $V \geq 0.000082$; and, assuming a space really may exist with this minimal volume, they give a qualitative explanation of the periodicity observed by Broadhurst *et al.* [26] in the distribution of galaxies. The present model is not small enough to account for that and similar periodicities [27], so we would like to see the discovery of CHMs much smaller than Weeks'.

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[12] Matrices *a* and *b* were adapted from Weeks' formalism (private communication). They are generators of a simpler presentation of Γ, and satisfy the relations $ab^{-1}ababab^{-1}a = ba^{-1}b^2a^{-1}baba = 1$. The γ_k's in the text satisfy similar relations, which are omitted here for brevity.

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[14] The map is an approximate equal area projection, based on axes X_1' parallel to the X_1 axes, with origin on Earth's position. Longitude increases from left to right, contrary to astronomical usage; but this is immaterial here, since the orientation of WM is arbitrary.

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