# Heterotic Parafermionic Superstring

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Superstrings have been postulated based on parafermionic partition functions which permit spacetime supersymmetry by generalized Jacobi identities. A comprehensive search finds new such identities. Quadrilateral anomaly cancellation gives constraints on allowed chiral fermions. Bosonic left movers and $Z_4$ parafermionic right movers combine in a new heterotic superstring, more constrained than the old one, yet equally applicable to physics.

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Progress in superstring theory has been slow during the last few years compared to its great strides about eight years ago. Part of the problem is that the most promising string theory, the heterotic string, has too much nonuniqueness in its low-energy predictions. This stems from the lack of calculational methods for selecting the correct ground state, for breaking supersymmetry, and for explaining the vanishing cosmological constant. None of these deep problems will be addressed here, but we adopt the viewpoint that the heterotic string *does* have too much freedom and that one should seek a more restrictive starting point.

Parafermionic strings have been considered for several years [1] and have recently been intensively studied in an important series of papers by the Cornell group [2–5]. The motivation is to reduce the critical spacetime dimension below $d_c=10$ by increasing the symmetry of the worldsheet. In the $d_c=10$ superstring there is worldsheet supersymmetry which pairs a boson with a fermion for each dimension; in a parafermionic superstring each boson is paired with a $Z_K$ parafermion thus realizing a worldsheet *fractional* supersymmetry [2].

In the general case, the left movers and right movers of a parafermionic string are assumed to carry $Z_K$ parafermionic fields of order $K_L$ and $K_R$, respectively, denoted $(K_L,K_R)$. The critical dimension for this theory is found to be $d_c=2+16/\max(K_L,K_R)$ except for the special case (1,1) which has $d_c=26$ [1,2]. The (1,1) bosonic string which is the simplest and oldest string theory has inadequacies such as a tachyon, no fermions, and no finiteness. Extending this model to a (2,2) superstring solves all these problems, but this type-II model has insufficient freedom to accommodate the low-energy physics of the standard model [6]. The (1,2) heterotic superstring has, on the other hand, perhaps too much freedom with respect to low-energy predictions.

The hope of parafermionic strings is to provide a more restrictive scenario for string model building. For $d_c \geq 4$ there are seven new parafermionic models, three with $K_R=4$ ($K_L=1,2,4$) and four with $K_R=8$ ($K_L=1,2,4,8$). In this Letter we shall examine these new models from the points of view of spacetime consistency (anomalies) and of possible connections to physics. Without knowledge of the fractional superconformal constraint algebra [5], we work only at the partition function level.

*Six-dimensional $K=4$ models.*— Models with $K_L \leq K_R=4$ have critical dimension $d_c=6$. In analogy with the type-II string, the (4,4) model is the most constrained of these three. The partition function for the (4,4) string, $Z_{(4,4)}$, is composed of $Z_4$ parafermionic string functions and was first written down in Ref. [2]. It is unique under the assumption that the individual components are tachyon-free (i.e., have a series expansion in non-negative powers of $q$).

Modular invariance itself does not fix the normalization of $Z_{(4,4)}$, denoted here by $\alpha$, but a physical interpretation of the partition function as counting states requires integer multiplicities, i.e., $16\alpha \in \mathbb{Z}$. The leading expansion of $Z_{(4,4)}$ gives $Z_{(4,4)}=\alpha(4-4)^2q^0+\cdots$ which is identified as four bosons and four fermions coming from each side of the string with total multiplicity $\alpha$. The standard interpretation of these states being a vector and spinor in the $d_c-2=4$ dimensional transverse space (thus forming a super-Maxwell multiplet) requires the stronger condition $\alpha \in \mathbb{Z}$.

Although the massless states can be identified as having usual bosonic and fermionic nature, the full spectrum contains states at mass level 0 and $\frac{1}{2}$ (mod1) of which the latter have well-known [3] difficulties with spin and statistics. Here we focus on the massless spectrum and assume the low-energy analysis is unaffected by the resolution of such difficulties encountered at the massive level.

With $\alpha=1$, the massless states are created by tensoring $d=6$ super-Maxwell multiplets on the left and right giving $N=4$A or 4B six-dimensional supergravity (in terms of $d=6$ symplectic Majorana-Weyl spinors). The $N=4$A theory is nonchiral because the spinors on the left and right have opposite chirality; $N=4$B is chiral and is prone to spacetime anomalies [7].

The massless type 4B string states form a $N=4$B graviton multiplet and tensor multiplet and the pure gravitational anomaly arising from this combination is nonvanishing [5,8]. In Ref. [5] it is proposed to cancel the anomaly by the judicious addition of twenty extra tensor multiplets. At the partition function level, these additional massless states can be accommodated by choosing the normalization $\alpha=6$. However, since it is not clear how these new physical states arise, we are led to consider the other possibility that they come from extra modular-invariant terms in $Z_{(4,4)}$ beyond those found in [2]; we shall examine this below.

Now consider the generalized heterotic (1,4) and (2,4) models. Similarly to the (1,2) heterotic string, the (1,4) model has a gauge group realized by a $c=26-d_c=20$ Kac-Moody algebra generated by the internal left-moving bosons. Massless states in this model arise from tensoring the left-moving spacetime vector and Kac-Moody currents with the super-Maxwell multiplet of right-movers to give six-dimensional $N=2$ supergravity coupled to super Yang-Mills theory.

In a general $N=2$ theory the possible multiplets are [8,9]

$$\begin{aligned}&\text{(a) } (e_\mu^a, \psi_{L\mu}^A, B_{\mu\nu}^{(-)}), \quad \text{(b) } (B_{\mu\nu}^{(+)}, \lambda_R^A, \phi)\,,\\ &\text{(c) } (A_\mu^a, \chi_L^{aA}), \quad \text{(d) } (\lambda_R^j, \phi^i)\,,\end{aligned} \tag{1}$$

where $i=1,\ldots,4n_d$, $j=1,\ldots,2n_d$, $\mu,\alpha=1,2,3,4$ are transverse space indices, and $A$ is in the **2** of Sp(2). The leading gravitational and gauge anomalies have the forms

$$\begin{aligned}I_8(R) &= \tfrac{1}{5760}(273n_a-29n_b+n_c-n_d)\,\mathrm{tr}R^4+\cdots\,,\\ I_8(F) &= \tfrac{1}{24}\,[\mathrm{Tr}F^4(\mathrm{adj})-\mathrm{Tr}F^4(\mathrm{matter})]\,,\end{aligned} \tag{2}$$

where the adjoint comes from the gauginos (c) and the matter comes from the gauge representations of (d). Since there is only one graviton, $n_a=1$. For the minimal $N=2$ matter content of the (1,4) model, we find that $n_b=1$, $n_c=\dim G$, and $n_d=0$, where $G$ is the gauge group. The gravitational anomaly is thus proportional to $(244+\dim G)$ and is hence nonzero for any $G$ of positive dimension.

Examining (2), we see that leading gravitational anomaly cancellation requires the addition of either new tensor matter (b) or hypermatter (d) multiplets. However, tensor matter multiplets cannot be created by tensoring bosonic left movers with any right-moving states. As a result, we must add $n_d$ hypermatter multiplets with $n_d=(244+\dim G)$ and in representations of $G$ such that the leading gauge anomaly also vanishes.

Maximal gauge symmetry for the (1,4) model in $d_c=6$ arises for $G=\mathrm{SO}(40)$ and requires $n_d=1024=2^{10}$ to cancel the $\mathrm{tr}R^4$ anomaly. These hypermatter states must further be put into representations of SO(40) so as to cancel the leading gauge anomaly. The smallest dimensional irreducible representations (irreps) of SO(40) are **1**,**40**,**780**,**819**,... with leading quadrilateral anomalies **0**,**1**,**32**,**48**,..., respectively [10]. We find that anomaly cancellation requires $n_{780}=1$, $n_1=244$, $n_{40}=n_{819}=0$ but this leads to a nonchiral model in $d=6$ since the hypermatter spinors can pair with the gauginos.

One *can* obtain chiral examples in $d=6$ with other choices of gauge group; as examples, we consider $\mathrm{SO}(24)\times\mathrm{SO}(16)$ and $\mathrm{SO}(24)\times E_8$ where the partition functions have been written in Ref. [3]. For the $\mathrm{SO}(24)\times\mathrm{SO}(16)$ model, SO(24) has irreps **1**,**24**,**276**,**299**,... with quadrilateral anomalies respectively **0**,**1**,**16**,**32**,.... SO(16) carries irreps **1**,**16**,**120**,**135**,... with anomalies **0**,**1**,**8**,**24**,.... Leading gravitational and gauge anomaly cancellation gives three conditions on the additional hypermatter representations, and we find the only possible irreps of the hypermatter that cancel the leading anomalies are [under (SO(24),SO(16))]

$$\begin{aligned}&(\mathbf{276},\mathbf{1})+(\mathbf{1},\mathbf{120})+244(\mathbf{1},\mathbf{1}) \text{ nonchiral}\,,\\ &(\mathbf{276},\mathbf{1})+8(\mathbf{1},\mathbf{16})+236(\mathbf{1},\mathbf{1}) \text{ chiral under SO(16)}\,,\\ &(\mathbf{1},\mathbf{120})+16(\mathbf{24},\mathbf{1})+136(\mathbf{1},\mathbf{1}) \text{ chiral under SO(24)}\,,\\ &16(\mathbf{24},\mathbf{1})+8(\mathbf{1},\mathbf{16})+128(\mathbf{1},\mathbf{1}) \text{ chiral under } \textit{both} \text{ SO(16) and SO(24)}\,.\end{aligned} \tag{3}$$

Cancellation of the nonleading and mixed anomalies will require a Green-Schwarz mechanism [11].

For $G=\mathrm{SO}(24)\times E_8$, a similar analysis shows that the irreps of the hypermatter which cancel leading anomalies are [under $(\mathrm{SO}(24),E_8)$]

$$\begin{aligned}&(\mathbf{276},\mathbf{1})+(\mathbf{1},\mathbf{248})+244(\mathbf{1},\mathbf{1}) \text{ nonchiral}\,,\\ &16(\mathbf{24},\mathbf{1})+(\mathbf{1},\mathbf{248})+136(\mathbf{1},\mathbf{1}) \text{ chiral under SO(24)}\,,\\ &(\mathbf{276},\mathbf{1})+492(\mathbf{1},\mathbf{1}) \text{ chiral under } E_8\,,\\ &16(\mathbf{24},\mathbf{1})+384(\mathbf{1},\mathbf{1}) \text{ chiral under } \textit{both} \text{ SO(24) and } E_8\,.\end{aligned} \tag{4}$$

Since the SO(40), $\mathrm{SO}(24)\times\mathrm{SO}(16)$, and $\mathrm{SO}(24)\times E_8$ partition functions given in [3] do not contain additional hypermatter multiplets, the chiral representations of (3) and (4) must arise from nonadjoint $G$ states coming in a novel way from the left-moving bosonic string. How these states may arise needs to be addressed if we wish to more fully understand six-dimensional anomaly cancellation.

The remaining six-dimensional parafermion model is the (2,4) one where the left movers are those of a superstring compactified from $d=10$ to $d_c=6$ while the right movers are the $K=4$ set already considered. The compactification of the left movers may be toroidal or more general. Compactification on a torus will give a $N=6$ supergravity theory which is anomalous.

A more general compactification of the left movers using, e.g., a free-fermion interpretation [12–14] can realize a $\hat{c}=10-d_c=4$ super Kac-Moody symmetry with gauge group $G$ of dimension up to 12 (in $d_c=6$). When this non-Abelian symmetry is realized on the left movers, no massless fermions arise from the left [6]. Thus the massless states form an $N=2$ theory similar to that of the (1,4) model. As before, the minimal partition function will have a leading gravitational anomaly which is nonvanishing but can be canceled by the addition of further massless states.

When considering the (4,4) model, we alluded to the necessity for additional massless states beyond those accommodated in the $\alpha=1$ partition function of [2]. We have therefore considered the use of $Z_4$ parafermionic string functions $c_n^l$ with odd $n$ [the string functions are related to the parafermionic characters by $\chi_n^l(q)=\eta(q)c_n^l(q)$; see, for example, [3,15,16]]. These odd functions did not play a role in the analysis of Ref. [3]. An exhaustive search for all tachyon-free modular-invariant (4,4) partition functions leads to fifteen independent choices of which four are vanishing and may be spacetime supersymmetric. One of these four is just $Z_{(4,4)}$, and the other three can be written as

$$\begin{aligned} Z_4^{(2)} &= (32|C_4^{\rm I}|^2+|D_4^{\rm I}|^2+32|E_4^{\rm I}|^2)-(32|C_4^{\rm II}|^2+|D_4^{\rm II}|^2+32|E_4^{\rm II}|^2)\,, \\ Z_4^{(3)} &= \overline{Z_4^{(4)}} = (32C_4^{\rm I}\overline{C_4^{\rm II}}+D_4^{\rm I}\overline{D_4^{\rm II}}+32E_4^{\rm I}\overline{E_4^{\rm II}})-(32|C_4^{\rm II}|^2+|D_4^{\rm II}|^2+32|E_4^{\rm II}|^2)\,, \end{aligned} \tag{5}$$

where $C_4^{\rm I}$, $D_4^{\rm I}$, and $E_4^{\rm I}$ are combinations of the even string functions

$$\begin{aligned} C_4^{\rm I} &= 2(c_0^2)^3c_2^4+(d_0^{0+})^3c_2^4+3d_0^{0+}(c_0^2)^2c_2^2\,, \\ D_4^{\rm I} &= (d_0^{0+})^4+8d_0^{0+}(c_0^2)^3-16(c_2^4)^4-16c_2^4(c_2^2)^3\,, \\ E_4^{\rm I} &= 4d_0^{0+}(c_2^4)^3+6c_0^2c_2^4(c_2^2)^2+d_0^{0+}(c_2^2)^3\,, \end{aligned} \tag{6}$$

and $C_4^{\rm II}$, $D_4^{\rm II}$, and $E_4^{\rm II}$ are combinations of the odd ones

$$\begin{aligned} C_4^{\rm II} &= 4(c_1^1)^3c_1^3+4c_1^1(c_1^3)^3\,, \\ D_4^{\rm II} &= (d_0^{0-})^4\,, \\ E_4^{\rm II} &= (c_1^1)^4+6(c_1^1)^2(c_1^3)^2+(c_1^3)^4\,, \end{aligned} \tag{7}$$

where $d_0^{0\pm}=c_0^0\pm c_0^4$.

Because these three new terms in (5) are modular invariant by themselves, they can be consistently added to the original (4,4) partition function $Z_{(4,4)}$. However, by examining the parafermionic string functions, we see that $C_4$, $D_4$, and $E_4$ only have states at mass level $\frac{5}{12}$, $\frac{2}{3}$, and $\frac{11}{12}$ (all mod 1), respectively. They hence contribute only massive states to the spectrum and have no effect on the low-energy properties of the (4,4) string.

The vanishing of the modular invariants, (5), arises from the new generalized Jacobi identities relating even and odd string functions,

$$C_4^{\rm I}=C_4^{\rm II},\quad D_4^{\rm I}=D_4^{\rm II},\quad E_4^{\rm I}=E_4^{\rm II}\,, \tag{8}$$

which we have proven on the basis of modular invariant function theory following the procedure given in [4].

We remark that the minus signs in (5) may be interpreted either as the statistics factor for fermions or as an internal projection. Without proper identification of spin and statistics, it is not possible to make this distinction. However, this issue is important since a projection actually *removes* physical states from the spectrum whereas the other case does not. Since the minus signs serve to project out the tachyons (present in $D_4$ and $E_4$), a natural interpretation is to view the signs as an internal projection. With this interpretation, there are no additional physical states arising from (5).

*Four-dimensional $K=4$ models.*— In compactifying the $K=4$ sector from $d_c=6$ to the physical spacetime $d=4$, since we have no complete understanding of the underlying worldsheet conformal field theory, it is impossible to make categoric statements about compactifications without any geometrical interpretation. This is a *caveat* of our analysis, but we feel confident that the rank of the $d=4$ gauge group cannot exceed that allowed in geometric compactification, namely, $r_{\rm max}=(d_c-4)$. Thus (4,4) compactified to $d=4$ can have an internal non-Abelian gauge group only with rank $r\leq 2$.

The (1,4) model in $d_c=6$ can have chiral fermions transforming under a rank $r=20$ gauge group [e.g., SO(24)×SO(16) or SO(24)×$E_8$ *ut supra*], and this will lead to an internal group with $r\leq 22$ in $d=4$. For example, the maximally symmetric case of (1,4) in $d=4$ is for gauge group SO(44) and is a nonchiral $N=2$ model with partition function easily constructed following the procedure given in [3]. In order to obtain a chiral model, one needs to break at least one of these supersymmetries to give $N\leq 1$ in $d=4$ dimensions.

For (2,4) models in $d=4$, because the left movers are governed by a super conformal field theory (SCFT), we can take the approach of Dixon, Kaplunovsky, and Vafa [6]. Since there are only two internal dimensions for the $K_R=4$ right movers, at least some of the rank 4 standard model gauge group would need to arise from the left movers. As long as a non-Abelian symmetry is realized, it then follows that no massless fermions can arise from the left movers [6]. Thus all massless four-dimensional states take the form of bosonic left-moving states tensored with (spacetime) supersymmetric right movers. If we assume that right-moving fermions are uncharged under right-moving symmetries, then any massless fermions must be right movers which are either gauge singlets or tensor producted with non-Abelian generators on the left. As a result, we demand the entire gauge group to be realized by a $\hat{c}=6$ super Kac-Moody algebra on the left.

We can now apply the result of [6] that if we seek $SU(3)\times SU(2)\times U(1)$ with a realistic fermion spectrum then we need $\hat{c}\geq 4+\frac{5}{3}+1=\frac{20}{3}$ which is too large to be accommodated by a $\hat{c}=6$ SCFT of the left movers. This argument is not as rigorous as the one used in [6] for the (2,2) type-II superstring because we have assumed that none of the gauge group arises from the $K_R=4$ right movers.

*Four-dimensional $K=8$ models.*— For $K=8$, the critical dimension is $d_c=4$ so that no compactification is necessary (or allowed). This is, however, a mixed blessing because the right movers for all $K_R=8$ models are in $N=1$ supermultiplets with fermion helicities $\pm\frac{1}{2}$ so that the tensor product with an independent $K_L=1$, 2, 4, or 8 sector will give nonchiral fermions. To achieve chirality would require a correlation between left and right movers which appears difficult without compactification.

In Ref. [2], one example of a tachyon-free modular-invariant spacetime supersymmetric (8,8) partition function is provided. We have confirmed that this partition function is unique by an exhaustive search of modular invariant combinations of all $K=8$ parafermion characters. Out of a total of twelve modular invariants, only two appropriate linear combinations are tachyon-free. However, the second tachyon-free combination is nonsupersymmetric so in this sense the (8,8) partition function is unique and, unlike the $K=4$ case above, we find no new $K=8$ identities beyond those of [4].

*Possible approaches to physics.*— Given the discussions of the present paper, it is clear that of all the new models the heterotic (1,4) model is the most promising for accommodating the low-energy physics of the standard model. For geometrically interpretable compactification on $M^4\times K^2$, however, there is an insuperable hurdle for four-dimensional chirality. Although for higher dimensions there exist Ricci-flat manifolds which can preserve spacetime supersymmetry, no such manifold exists for $K^2$ except a torus which leads to a nonchiral $N=2$ model. Hence we require a *nongeometric* reduction to $d=4$ from the (1,4) string in $d_c=6$. This is precisely what has been suggested based on the independent consideration of spin and statistics by the Cornell group [3,5]. The (1,4) parafermionic heterotic superstring, quite unlike the less constrained and much more familiar (1,2) heterotic superstring, may thus exist consistently in $d=4$ *only*, and not in the critical dimension; chiral fermions must then arise from $N=1$ right movers in a complex representation of the gauge group generated by the left movers.

The status of the mathematical consistency of the (1,4) parafermionic superstring is yet to be understood at the level of the older (1,2) heterotic or (2,2) type-II superstring. Construction of the worldsheet constraint algebra for the $K=4$ parafermionic superstring is thus an interesting issue and merits further study.

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