# A Spherical Impulse Gravity Wave

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An exact solution of Einstein's vacuum field equations describing a spherical-fronted impulsive gravity wave is given in a coordinate system in which the metric tensor is continuous across the (future null cone) history of the wave front in Minkowskian space-time. A coordinate transformation is presented which reduces the line element to Minkowskian form in the future of the null cone (it is manifestly in Minkowskian form in the past of the null cone) and enables us to demonstrate how the solution has been constructed using the Penrose procedure of subdividing flat space-time into two halves with the future null cone as boundary and then reattaching the halves with a warp.

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The spherical-fronted impulsive gravity wave is more difficult to construct than a plane-fronted impulsive wave for the following reason: The general Petrov type-$N$ spherical wave solutions of Einstein's vacuum field equations are described by the Robinson-Trautman [1,2] solutions of type $N$. These solutions depend upon one real-valued function $q$ which (just as in the plane-fronted case) is harmonic in the "angular" coordinates and has an arbitrary dependence on a null coordinate $v$. The histories of the wave fronts are the null hypersurfaces $v=\text{const}$ and are future "null cones" in the sense that they are generated by expanding, shear-free null geodesics. To construct a spherical-fronted impulsive wave as a spherical case the "obvious" thing to do is to require $q$ to have a Dirac delta function dependence on $v$ with singular value corresponding to, say, $v=0$. The curvature tensor will then have this delta function behavior typical of an impulsive gravity wave. The metric tensor, however, turns out to be quadratic in the delta function and this is not acceptable mathematically. The way out of this problem has been given by Penrose [3] in a classical paper on the geometry of impulsive gravity waves. Taking the flat space-time line element in the form

$$ds^2=2u^2\,d\zeta\,d\bar{\zeta}+2du\,dv \tag{1}$$

(here $\zeta$ is a complex coordinate with complex conjugate $\bar{\zeta}$ and $u,v$ are real coordinates), the hypersurfaces $v=\text{const}$ are future null cones. Subdivide the space-time $M$ into two halves using the null cone $v=0$ (say). The halves are then $M^+$ $(v>0)$ and $M^-$ $(v<0)$ and are reattached with the identification (a "warp")

$$(\zeta,\bar{\zeta},u,v=0)_{M^-}=\left[h(\zeta),\bar{h}(\bar{\zeta}),\frac{u}{|h'(\zeta)|},v=0\right]_{M^+}, \tag{2}$$

where $h$ is an arbitrary analytic function of $\zeta$ and $h'=dh/d\zeta$. The choice (2) is made so as to have the metric induced on $v=0$ from $M^-$ agree with the metric induced on $v=0$ from $M^+$. The result, following from Penrose's general theory, is that $v=0$ is now the history of a spherical impulsive gravity wave, the space-time has vanishing Ricci tensor, and the curvature tensor is Petrov type $N$ with delta function dependence on $v$ (singular at $v=0$). To complete the picture one would like to show explicitly the metric tensor of the space-time continuous across $v=0$. As Penrose states, "This can in fact be seen by a transformation to a continuous coordinate system, but this is complicated." It is this calculation which we give explicitly below. The result is in a form which seems to be simpler than that envisaged by Penrose.

To avoid confusion it is convenient from now on to write the equation of the history of the spherical impulsive gravity wave as $V=0$. We have found that in a coordinate system $(Z,\bar{Z},U,V)$ in which the metric tensor is continuous across $V=0$ the line element of the space-time is given by

$$ds^2=2U^2\left|dZ+\frac{V\vartheta(V)}{2U}\bar{H}\,d\bar{Z}\right|^2+2dU\,dV. \tag{3}$$

Here $H$ is an arbitrary analytic function of the complex coordinate $Z$ (complex conjugation being denoted by a bar) and $\vartheta(V)$ is the Heaviside step function which is equal to unity if $V>0$ and equal to zero if $V<0$. Direct calculation from (3) shows that the corresponding Ricci tensor vanishes for all $V$ and thus (3) represents a solution of Einstein's vacuum field equations. The only non-identically vanishing component of the curvature tensor for (3) is (in Newman-Penrose [4] notation)

$$\Psi_4=U^{-1}H(Z)\delta(V), \tag{4}$$

where $\delta$ is the Dirac delta function. Thus the curvature tensor is Petrov type $N$ with degenerate principal null direction given by the vector field $\partial/\partial U$ evaluated on $V=0$. In general the integral curves of the vector field $\partial/\partial U$ are null geodesics with complex shear

$$\sigma=-\tfrac{1}{2}\phi^{-1}HV\vartheta(V), \tag{5}$$

with

$$\phi=U^2-\tfrac{1}{4}V^2\vartheta(V)|H|^2, \tag{6}$$

and (real) expansion

$$\rho=\phi^{-1}U . \tag{7}$$

We note that on $V=0$ we have $\sigma=0$, $\rho=U^{-1}$ and the integral curves of $\partial/\partial U$ generate a future null cone which is the history of a spherical-fronted impulsive gravity wave.

The line element (3) for $V<0$ and for $V>0$ can be written in the Minkowskian form (1). For $V<0$ this is given trivially by

$$\zeta=Z,\ \ \bar{\zeta}=\bar{Z},\ \ u=U,\ \ v=V . \tag{8}$$

For $V>0$ the transformation leading to (1) is

$$\zeta=h(Z)+\frac{V}{2U}\alpha\bar{\beta}\left[1-\frac{V}{4U}|\beta|^2\right]^{-1}, \tag{9a}$$

$$\bar{\zeta}=\bar{h}(\bar{Z})+\frac{V}{2U}\bar{\alpha}\beta\left[1-\frac{V}{4U}|\beta|^2\right]^{-1}, \tag{9b}$$

$$u=\frac{U}{|\alpha|}\left[1-\frac{V}{4U}|\beta|^2\right], \tag{9c}$$

$$v=|\alpha|V\left[1-\frac{V}{4U}|\beta|^2\right]^{-1}. \tag{9d}$$

Here $h$ is an arbitrary analytic function of $Z$, $\alpha=dh/dZ$, $\alpha\beta=d^2h/dZ^2$, and the function $H$ in (3) is obtained from

$$H=\frac{d\beta}{dZ}-\frac{1}{2}\beta^2 . \tag{10}$$

Approaching $V=0$ from $V<0$ and from $V>0$ we see that the transformations (8) and (9) incorporate Penrose's geometrical interpretation of reattaching the two halves of Minkowskian space-time on the null cone $V=0$ with the identification

$$(Z,\bar{Z},U,V=0)_{M^-}=\left[h(Z),\bar{h}(\bar{Z}),\frac{U}{|\alpha|},V=0\right]_{M^+} . \tag{11}$$

Writing $Z=x^1$, $\bar{Z}=x^2$ we see from (3) that the metric tensor components $g_{11}$, $g_{12}$ have the properties that $\partial g_{12}/\partial V$ is continuous across $V=0$ while there is a jump in $\partial g_{11}/\partial V$ and $\partial^2 g_{12}/\partial V^2$ across $V=0$. From the Penrose theory there is an elegant geometrical explanation for this behavior: Let $C$ be a null geodesic congruence in Minkowskian space-time intersecting $V=0$ (and with tangent vector normalized to have scalar product of $-1$ with $\partial/\partial U$). Then $V=0$ is the history of an impulsive gravity wave if the *convergence* of $C$ is continuous across $V=0$ (this condition corresponds to the vanishing of the Ricci tensor in the present case) and the curvature tensor has a Petrov type-$N$ delta function (singular on $V=0$) if the *shear* of $C$ is discontinuous across $V=0$. In the example above, the convergence of $C$ is continuous across $V=0$ because $\partial g_{12}/\partial V$ is continuous across $V=0$ and there is a jump in the shear of $C$ because $\partial g_{11}/\partial V$ is discontinuous across $V=0$. Now the propagation equation for the convergence of $C$ along $C$ ensures that the rate of change of the convergence of $C$ along $C$ is discontinuous since the shear is discontinuous. This rate of change of the convergence of $C$ along $C$ is proportional to $\partial^2 g_{12}/\partial V^2$ and thus this quantity jumps across $V=0$.

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