Quantum Railroads: Introducing Directionality to Anderson Localization

C. Barnes, B. L. Johnson, and G. Kirczenow

Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

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We present a theory of electron transport in novel disordered waveguides that support different numbers N and M of modes propagating in opposite directions, i.e., "quantum railroads" (QRR). Anderson localization and the integer quantum Hall effect are special cases of our theory. More generally, our analytic results based on scattering matrix theory and our numerical simulations show that disorder results in *directed* localization in QRR's: The limiting transmittance of a macroscopic QRR is $T=|N-M|$ in the majority direction but $T'=0$ in the minority direction.

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It was first proposed by Anderson [1] that under suitable conditions waves may become spatially localized in the presence of disorder. This remarkable insight has stimulated a great deal of theoretical and experimental interest in a broad class of problems [2] referred to collectively as "Anderson localization." As was pointed out by Mott and Twose [3], in disordered one-dimensional systems every eigenstate is localized to some region of space. As a consequence, at zero temperature, the conductance of a quasi-one-dimensional waveguide with random elastic scatterers decreases on the average to zero as the length of the waveguide increases. This is despite the strong fluctuations which occur. For weakly disordered systems these fluctuations have the universal feature that the variance of the conductance is a constant $\sigma_g^2=(e^2/h)^2$. This has been demonstrated theoretically by Al'tshuler [4] and Lee, Stone, and Fukuyama [5] and experimentally by Blonder, Dynes, and White [6] and Umbach *et al.* [7] and Webb *et al.* [8]. More recently Pendry, MacKinnon, and Pretre [9], and MacKinnon [10] have discussed conductance fluctuations for more general circumstances and their findings suggest that for long strongly localized systems they occur rarely but with maximal strength.

An essential, and at first sight quite general, assumption that is implicit in all of the theoretical work on one-dimensional Anderson localization is that the waveguide under consideration supports equal numbers of modes propagating in opposite directions. However, not all quasi-one-dimensional systems have this property, as has become increasingly apparent following the discovery of the quantum Hall effect by von Klitzing, Dorda, and Pepper [11]. Laughlin [12] explained the observed quantization of the Hall conductance of the 2D electron gas in semiconductor heterostructures in integer multiples of e^2/h [11], using a gauge invariance argument. Subsequently, however, Streda, Kucera, and MacDonald [13], Jain and Kivelson [14], and Büttiker [15] proposed an alternate point of view, in which the quantum Hall effect is explained on the basis of the Landauer [16] theory of one-dimensional transport, within the framework of magnetic edge states introduced by Halperin [17]. These edge states, which derive from the quantized Landau levels of the 2D electron gas (2DEG) in strong magnetic fields, follow the edges of the sample, and are the one-dimensional transport channels (or modes) of these theories. This edge-state picture of the integer quantum Hall effect has gained wide acceptance.

Although in the quantum Hall effect transport is effectively one dimensional, the considerations leading to 1D Anderson localization [1-7,9,10] do not apply to it because *all* of the channels at a given edge of the sample propagate in the *same* direction, which, in macroscopic systems, makes backscattering of electrons impossible in the quantum Hall regime. This important point was elucidated by Büttiker [15]. Thus there can be *perfect* transmission of electrons through a disordered macroscopic sample in the quantum Hall regime, in stark contrast to the zero average transmission found in more conventional 1D systems.

The purpose of this Letter is to introduce a new physical phenomenon—directed localization. We report on a theoretical study of the general problem of quasi-one-dimensional transport in a disordered waveguide which can support *arbitrary* numbers N and M of modes propagating in the two opposite directions. Our waveguide may be visualized as a railroad connecting Los Angeles and New York with trains running in the direction from LA to NY on N sets of tracks, and from NY to LA on another M sets. At any time, a train may switch from one track to another, changing its direction of travel or not, depending on the tracks between which it switches. Since the "trains" represent propagating waves (or quantum particles such as electrons), and "switching tracks" is a quantum scattering process, we will speak of a "quantum railroad" (QRR). One can then ask, if a train sets out from LA, what is the probability of it eventually arriving in New York? The two cases $N=M$ and $M=0$ ($N\neq0$) correspond, respectively, to the case of 1D Anderson localization and to the case of perfect transmission of edge channels in the quantum Hall effect discussed above. We wish to investigate the general case, which has not been explicitly studied in the context of localization, i.e., we find that if $N\neq M$, the QRR exhibits a novel behavior that we will refer to as "directed localization." For example if $N > M$ then a train that leaves NY never arrives in LA—it always returns to NY; whereas a train leaving LA has a finite probability of arriving in NY. This probability approaches a nonzero limiting value as the number of scattering events between LA and NY increases, i.e., the QRR eigenstates are asymmetric. They behave as if they were localized in so far as transmission from NY to LA is concerned, but a significant fraction of them are extended in the opposite direction, from LA to NY. We will refer to this phenomenon as directed localization.

At present, to our knowledge, no example of the general QRR that we consider here has been realized experimentally. However, in the presence of magnetic fields, periodic 2DEG structures such as the Azbel-Wannier-Hofstadter (AWH) system [18–20], which have been investigated experimentally by Gerhardts, Weiss, and Wulf [21], and 2D arrays of coupled quantum dots [22–26], have all been predicted theoretically [23,24,27,28] to exhibit multiple edge channels with different numbers of modes propagating in opposite directions when the system has a finite width. They are thus potential experimental realizations of the general QRR. The Hall conductance of the AWH system with no edges is predicted by consideration of the Kubo formula and the topology of the state structure [29] (see Aoki [30]) and is in agreement with the edge-state picture of Ramal *et al.* [27] and MacDonald [28] in which the conductance is the algebraic sum of edge states in an AWH system of finite width.

The transport properties of a QRR are defined by a scattering matrix of the form

$$\mathbf{S}=\begin{pmatrix}\mathbf{T} & \mathbf{R}\\ \mathbf{R}' & \mathbf{T}'\end{pmatrix}, \tag{1}$$

where $\mathbf{T}$ is an $N\times N$ transmission matrix containing the complex *amplitudes* for scattering between the N forward modes, $\mathbf{T}'$ is the $M\times M$ transmission matrix for the M reverse modes, and $\mathbf{R}$ and $\mathbf{R}'$ are the corresponding reflection matrices. The two-terminal transmittances T, T' and reflectances R,R' of the system, when connected at either end to perfectly emitting and absorbing reservoirs, are simply related to the elements of this $\mathbf{S}$ matrix through the norms of the transmission and reflection matrices [31–33]

$$T=\|\mathbf{T}\|^2,\quad T'=\|\mathbf{T}'\|^2,\quad R=\|\mathbf{R}\|^2,\quad R'=\|\mathbf{R}'\|^2. \tag{2}$$

In this case the norm is defined by the inner product $(\mathbf{a},\mathbf{b})=\mathrm{trace}(\mathbf{ab}^\dagger)$, where $\mathbf{a}$ and $\mathbf{b}$ are matrices with suitable dimensions.

From these definitions it is easy to see that in a waveguide containing no scattering the transmittances will have the form $T=N$ in the forward direction and $T'=M$ in the reverse direction.

If we now introduce a series of unitary scatterers into the QRR then the scattering matrix of the whole system must also be unitary. This condition, given by $\mathbf{SS}^\dagger=\mathbf{S}^\dagger\mathbf{S}=\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, implies severe constraints on the quantities given in (2) above. For instance, for the diagonal blocks we find, $\mathbf{TT}^\dagger+\mathbf{RR}^\dagger=\mathbf{I}$ and $\mathbf{T}^\dagger\mathbf{T}+\mathbf{R}'^\dagger\mathbf{R}'=\mathbf{I}$. Taking traces gives $T+R=N$ and $T+R'=N$. These results, together with those obtained from the off-diagonal blocks, give the relations

$$T+R=N,\quad R=R',\quad T'=T-(N-M). \tag{3}$$

These three relations together with the fact that T,T', R,R' are real and positive imply that any such system will have transmittances in the ranges

$$0\le T'\le M, \tag{4}$$

$$N-M\le T\le N. \tag{5}$$

Hence, we see that the transmittance in the majority channel direction has a lower limit of $T=N-M$ and in the minority direction of $T'=0$.

The second part of our proof is to make a justification for asserting that as we make our QRR longer both T and T' will tend to decrease and therefore that for a typical system they will eventually reach their minimum values. Adding a strip of material containing scatterers to a QRR will cause a change in its reflection matrix given by

$$\mathbf{R}_{+1}=\mathbf{R}+\mathbf{B}, \tag{6}$$

where

$$\mathbf{B}=\mathbf{T}\mathbf{r}(1-\mathbf{R}'\mathbf{r})^{-1}\mathbf{T}'. \tag{7}$$

Capitals represent the transmission and reflection matrices of the initial QRR and lower case those of the added slice. Taking the norm of Eq. (6) and using Eqs. (3) we find

$$T_{+1}=T-\|\mathbf{B}\|^2-(\mathbf{B},\mathbf{R})-(\mathbf{R},\mathbf{B}). \tag{8}$$

Note that the same relation holds for T'. Since the last two terms in Eq. (8) may be positive or negative it is clear that adding an extra slice of disordered material can cause the transmittance to increase or decrease. However, the possible choices for making T_{+1} larger are limited in comparison to those which make it smaller and also the range of those choices is highly dependent on the details of the $\mathbf{S}$ matrix of the initial QRR. Thus unless the QRR is made longer by adding material with scattering properties highly correlated to what has come before, the transmittance will tend to decrease. For example, if we construct a QRR from a stack of uncorrelated random unitary scatterers, because there are no correlations between the reflection phases of the added slice it is easy to show that $\overline{(\mathbf{R},\mathbf{B})}=0$ and $\overline{(\mathbf{B},\mathbf{R})}=0$ for each added slice. The bar indicates an average over all possible slices which may be added. Both inner products may be expanded as multinomial series in the reflection amplitudes contained in $\mathbf{r}$. After averaging, these series become expansions in the multivariate moments of the elements of **r**. Each of these moments is zero if there are no correlations between the reflection phases and therefore on average the inner products are zero. This fact indicates that on average making the QRR longer will reduce its transmittance in either direction. In order to investigate this point and make it clearer for ourselves we carried out a number of numerical simulations for different values of N and M. The basis of our numerical method was to generate random unitary scattering matrices using Gram-Schmidt orthogonalization on a set of $N+M$ random vectors. We then convert these into transfer matrices with the correct symmetries and take products to find the transfer matrix for the whole system. This matrix may then be solved for the transmission matrices and the reflection matrices and the transmittances and reflectances found from their norms. Figure 1 shows two simple cases for comparison. $N=M=2$, the case where Anderson localization sets in, and $N=2, M=1$ a QRR case. Both plots show that both transmitivities reduce to their minimum values with increasing length despite quantum L fluctuations. Note that the transmission falls exponentially to its minimum value in both cases—the general result for quantum transport in 1D.

If adding an extra slice of disordered material to a QRR typically reduces its transmittance in both directions, then for long systems experiments will measure the values

$$T'=0, \tag{9}$$

$$T=N-M. \tag{10}$$

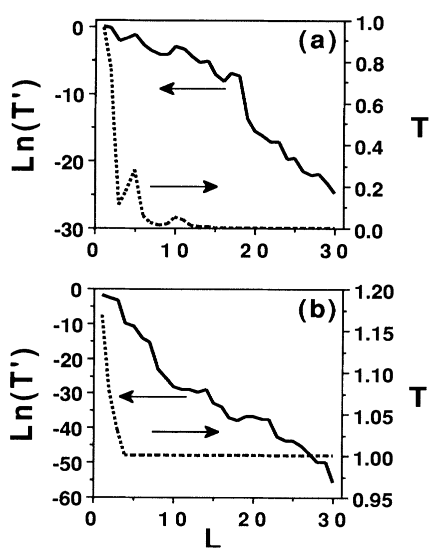

FIG. 1. (a) Forward (T) and reverse (T') transmittivities of a QRR as a function of system length L for $N=M=2$; (b) same as in (a) except $N=2$ and $M=1$. Note that the transmission exhibits exponential decay and fluctuations with length typical of quantum conduction in disordered systems.

The possibility that they may reduce to different minima is excluded by the fact that $\mathbf{B}=\mathbf{0}$ typically only if $\mathbf{T}'=\mathbf{0}$. The QRR, therefore, is a system in which the transmittance is directed. In one direction the system is opaque and in the other it has a transmittance which tends to a quantized value with increasing length. Fluctuations in the transmittance can of course play an important role acting like pinholes through an otherwise opaque system. Samples like these, while being of great interest to theorists, are not typical [9,10].

There are ways in which the directed transmittance of a QRR may be disturbed. The N forward modes and M reverse modes of the QRR may not be the only modes in the system. In general there could be other propagating modes and evanescent modes. We wish to emphasize that we have been considering the case of dilute scatterers, i.e., we imagine a waveguide in which the eigenmodes are intact, and the scattering events serve to occasionally mix the modes (unitary "switches" in the railroad). Under these conditions, the evanescent modes cannot connect the ends of a macroscopic sample, and hence do not destroy the effect. (Unitarity implies that an electron entering the system in a propagating mode cannot become permanently trapped in evanescent modes.) However, if scattering into other propagating modes in the system occurs, whether via evanescent modes or directly, it will alter the predicted effects, since in general it will change the effective values of N and M. In the case of an AWH system, this would correspond physically to scattering an electron from one edge of the sample to the opposite edge, a negligibly improbable process for macroscopically wide samples if the Fermi energy lies in a bulk spectral gap. Another possibility is that temperature or other inelastic processes may disturb the mode structure so that from one scattering event to the next there are different numbers of available modes. This will of course destroy the effect.

In summary, we have introduced quantum railroads, a general category of disordered systems that includes the one-dimensional structures that support Anderson localization, as well as the perfectly conducting edge channels that characterize the quantum Hall effect. We have shown that QRR's exhibit a novel type of localization that is directional in its nature. We find that the transmission is given by $N-M$ in the majority direction and 0 in the minority direction; this result corresponds to that obtained previously for the AWH model by topological arguments.

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