Spherical Model for Turbulence

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We develop a large-N method for the problem of homogeneous turbulence. The spherical ($N\rightarrow\infty$) limit yields Kraichnan's direct interaction approximation equations. Implications for real turbulence ($N=1$) are discussed. In particular, we argue that the renormalization-group results obtained by setting the expansion parameter $y=4$ are incorrect, and that the Kolmogorov exponent ζ has a nontrivial dependence on N, with $\zeta(N\rightarrow\infty)=\frac{3}{2}$. This value is remarkably close to the experimental result $\zeta\simeq\frac{5}{3}$, which must therefore result from higher-order corrections in powers of $1/N$.

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Possibly the single most outstanding problem in the theory of homogeneous turbulence in fluids is a detailed understanding of the turbulent energy cascade, commonly known as the Kolmogorov cascade. Thus if we consider a fluid that is stirred at some very large length scale, m_0^{-1}, while dissipation due to viscous processes occurs only on very small length scales $\Lambda^{-1}\ll m_0^{-1}$, then in the *inertial range* $m_0\ll k\ll\Lambda$ we expect power-law behavior for the energy spectrum, $E(k)\approx Ck^{-\zeta}$, with a characteristic universal exponent (the Kolmogorov exponent) ζ. Experimentally it is found that $\zeta\simeq\frac{5}{3}$ [1], in amazing agreement with the original Kolmogorov argument [2]. Many subsequent treatments, based to varying degrees on the actual fluid equations [3,4], have obtained the $\frac{5}{3}$ law as well. To date, however, there is no theoretical evidence that this law is exact. The danger with existing derivations is that, very much like the equivalence of all mean-field theories in critical phenomena, they may simply be complicated rephrasings of Kolmogorov's original dimensional analysis.

What makes turbulence such a difficult problem is, first, its nonequilibrium nature, and, second, the lack of a small expansion parameter. The most successful method in critical phenomena, the epsilon expansion around dimension $d=4$, does have an analog in turbulence theory, but is completely uncontrolled (see below).

We turn, therefore, to the second general method used in critical phenomena, namely, the $1/N$ expansion. The idea here is to generalize the model under consideration to one with a higher symmetry, indexed by the integer N, and consider the limit where N is large. In standard applications to critical phenomena [5], N is the dimension of the rotation group, $O(N)$, and physical values are $N=1,2,3$; the limit $N\rightarrow\infty$ yields the exactly soluble *spherical model*, and a systematic expansion in powers of $1/N$ may be developed [5]. The main advantage of the approach is that the dimensionality d becomes a free variable.

To develop an analogous approach in the theory of turbulence, we consider the following generalization of the Navier-Stokes equations for an incompressible, stochastically driven fluid (sums over repeated indices are understood):

$$\frac{\partial\mathbf{v}^\alpha}{\partial t}+\lambda_0A_N^{\alpha\beta\gamma}(\mathbf{v}^\beta\cdot\nabla)\mathbf{v}^\gamma=-\nabla p^\alpha+\nu_0\nabla^2\mathbf{v}^\alpha+\mathbf{f}^\alpha, \quad (1a)$$

$$\nabla\cdot\mathbf{v}^\alpha=0,\quad \alpha=1,\ldots,N, \quad (1b)$$

where $\mathbf{v}^\alpha(\mathbf{x},t)$ are velocity fields; $p^\alpha(\mathbf{x},t)$ are pressures, determined by (1b); ν_0 is the viscosity; and $\mathbf{f}^\alpha(\mathbf{x},t)$ are independent random stirring forces, taken to be Gaussian with zero mean and Fourier-transformed variance

$$\langle f_i^\alpha(\mathbf{k},\omega)f_j^\beta(\mathbf{k}',\omega')\rangle=D(\mathbf{k},\omega)\tau_{ij}(\mathbf{k})\times\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta_{\alpha\beta}, \quad (2)$$

where $\tau_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ is the usual transverse projection operator. In order to impose a higher symmetry on these equations, we assume them to be invariant under some group of transformations

$$\mathbf{v}'^\alpha=D_{\alpha\beta}^N(u)\mathbf{v}^\beta,\quad u\in G_N, \quad (3)$$

where the $N\times N$ matrices $\mathbf{D}^N(u)$ form an irreducible unitary representation of the group G_N. This requires that the rank-3 tensor $\mathbf{A}_n$ obey

$$A_N^{\alpha\beta\gamma}=D_{\alpha\alpha'}^N(u)D_{\beta\beta'}^N(u)D_{\gamma\gamma'}^N(u)A_N^{\alpha'\beta'\gamma'},\quad \forall u\in G_N, \quad (4)$$

i.e., that it be invariant under the group G_N. Amit and Roginsky [6] have considered N-component generalizations of the Potts model, which also requires a cubic invariant. They chose the group $G_N=O(3)$ for all N, but allowed the *dimension* of the representation to diverge with N [7]: $N=2l+1$, where the total angular momentum index l is taken to be even (see below). Thus the matrices $\mathbf{D}^N$ are the famous d matrices from quantum mechanics [8]. The cubic invariants $A_N^{\alpha\beta\gamma}$ are then the Wigner $3j$ symbols [9]

$$f(l)^{-1}A_N^{m_1m_2m_3}\equiv(-1)^{m_1}\begin{pmatrix} l & l & l\\ -m_1 & m_2 & m_3\end{pmatrix}=\frac{1}{\sqrt{2l+1}}\langle lm_1|lm_2lm_3\rangle \quad (5)$$

with the familiar Clebsch-Gordan coefficients on the right-hand side; $f(l)$ is a normalization to be chosen below. In applying this choice to the turbulence problem it is most convenient to use a complex notation for the velocities, with the constraints that $\mathbf{v}^{-\alpha}=(-1)^{\alpha}\mathbf{v}^{\alpha *}$, $\mathbf{f}^{-\alpha}=(-1)^{\alpha}\mathbf{f}^{\alpha *}$ [10].

We note that any set of equations of the form (1) possesses a generalization of Galilean invariance. Let $\tilde{A}_N^{\beta\gamma}=\sum_\alpha A_N^{\alpha\beta\gamma}$, and let g^β be any left eigenvector of $\tilde{\mathbf{A}}_N$, with corresponding eigenvalue μ: $\tilde{A}_N^{\beta\gamma}g^\beta=\mu g^\gamma$. Then the equation of motion for the *averaged* velocity, $\mathbf{v}=(1/N)\times\sum_\alpha \mathbf{v}_\alpha$, is invariant under the transformation

$$\mathbf{v}'^{\alpha}(\mathbf{x},t)=\mathbf{v}^{\alpha}(\mathbf{x}+\lambda_0 g^\alpha \mathbf{v}_0 t,t)-\frac{1}{\mu}g^\alpha \mathbf{v}_0,\quad \alpha=1,\ldots,N\,, \tag{6}$$

where $\mathbf{v}_0$ is an arbitrary fixed velocity. Since there are N eigenvectors g^β, this yields N distinct Galilean-type symmetries [11].

We now turn to the $N\to\infty$ limit, relying heavily on Amit and Roginsky's [6] investigations into the behavior of the $A_N^{\alpha\beta\gamma}$ for large l. Since the topology of the diagrams in the perturbation theory expansion for their Potts model is identical to ours, their elucidation of the surviving diagrams when $N\to\infty$ can be transferred with little change to our problem. The appropriate normalization in (5) is found to be $f(l)=\sqrt{2l+1}=\sqrt{N}$. There are two important correlation functions: the velocity-velocity correlation function U and the response function G defined by

$$\langle v_i^\alpha(\mathbf{k},\omega)v_j^\beta(\mathbf{k}',\omega')\rangle=U(\mathbf{k},\omega)\tau_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta_{\alpha\beta}\,, \tag{7a}$$

$$\langle f_i^\alpha(\mathbf{k},\omega)v_j^\beta(\mathbf{k}',\omega')\rangle=G(\mathbf{k},\omega)D(\mathbf{k},\omega)\tau_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta_{\alpha\beta}\,. \tag{7b}$$

The average is over the ensemble of forcing functions $\mathbf{f}^\alpha$ [12]. The energy spectrum is given by $E(k)=k^{d-1}\int(d\omega/2\pi)U(k,\omega)$. The basic result is that when $N\to\infty$, only graphs with the maximum number of bubbles at each order survive [10,13], each with unit coefficient. These diagrams may be resummed exactly, and we obtain two nonlinear coupled integral equations for G and U:

$$1/G(\mathbf{k},\omega)=-i\omega+\nu_0 k^2+\lambda_0 k^2\int_q\int_\Omega b(\mathbf{k},\mathbf{q})U(\mathbf{k}-\mathbf{q},\omega-\Omega)G(\mathbf{q},\Omega)\,, \tag{8a}$$

$$U(\mathbf{k},\omega)=|G(\mathbf{k},\omega)|^2\left[D(\mathbf{k},\omega)+\lambda_0 k^2\int_q\int_\Omega a(\mathbf{k},\mathbf{q})U(\mathbf{k}-\mathbf{q},\omega-\Omega)U(\mathbf{q},\Omega)\right], \tag{8b}$$

where $b(\mathbf{k},\mathbf{q})=P_{ijl}(\mathbf{k})\tau_{jm}(\mathbf{k}-\mathbf{q})P_{lmi}(\mathbf{q})/(d-1)k^2$, $a(\mathbf{k},\mathbf{q})=\frac{1}{2}[b(\mathbf{k},\mathbf{q})+b(\mathbf{k},\mathbf{k}-\mathbf{q})]$, and $P_{ijl}(\mathbf{k})=\tau_{ij}(\mathbf{k})k_l+\tau_{il}(\mathbf{k})k_j$. We use the convenient notation $\int_q\equiv\int d^dq/(2\pi)^d$, etc. These equations are very well known in the theory of turbulence: They are Kraichnan's direct interaction approximation (DIA) equations [14], originally derived as the lowest in a hierarchical closure scheme [15].

We now turn to a discussion of the properties of (8) and their implications for the energy cascade. We first contrast (8) with the renormalization-group approach [4] in which one considers driving spectra of the form $D(\mathbf{k},\omega)\equiv D(k)\sim k^{4-d-y}$, $k\to 0$ ($y=2-d$ corresponding to thermal equilibrium). For $y<0$ it is found [4] that the nonlinearity λ_0 scales to zero at long wavelengths, and the functions (7) take forms characteristic of linear hydrodynamics: $G(\mathbf{k},\omega)\approx(-i\omega+\nu_R k^2)^{-1}$, $U(\mathbf{k},\omega)\approx D(k)/(\omega^2+\nu_R^2k^4)$, $k\to 0$, where ν_R is the renormalized, large length-scale (eddy) viscosity. For $y>0$ the nonlinearity remains finite on large length scales, the functions G and U are strongly renormalized, and nontrivial *scaling laws* develop [4]

$$G(\mathbf{k},\omega)\approx A_1 k^{-z}g(\omega/\tilde{\nu}k^z)\,, \tag{9a}$$

$$U(\mathbf{k},\omega)\approx A_2 k^{-\Delta}u(\omega/\tilde{\nu}k^z),\quad k,\omega\to 0\,, \tag{9b}$$

where the exponents Δ,z and the scaling functions g,u are universal, while A_1, A_2, and $\tilde{\nu}$ are nonuniversal scale factors. The energy exponent is given generally by $\zeta=\Delta-z-d+1$. The renormalization-group formalism allows one to calculate z, Δ to *all orders* in y [4], with the result that $z=2-y/3$, $\Delta=d+y/3$, and hence $\zeta=2y/3-1$. Now, real turbulence corresponds to $D(k)\to 0$ outside some narrow range $k\lesssim m_0$, i.e., in effect $y\to\infty$ (we call this "short-ranged driving"). However, it was observed [4(b)] that when $y=4$ one has $\zeta=\frac{5}{3}$. This, together with the fact that $y=4$ corresponds to a natural boundary in the renormalization-group formalism (beyond which an infinite family of new relevant operators appears) [4(b)] led to the speculation that for $y>4$ the exponent values should *stick* at their $y=4$ values. Precisely this behavior occurs in standard critical phenomena where the mean-field fixed point takes over for $d>4$. Although there is no direct evidence for this idea, much subsequent work has been based on its assumed validity [16].

We shall now show that the $1/N$ expansion yields a rather different scenario. To motivate it we consider an analogy to magnets with power-law interactions [17]. Consider an Ising model with Hamiltonian $H=-\sum_{i\neq j}J_{ij}s_is_j$, with $J_{ij}\sim|\mathbf{R}_i-\mathbf{R}_j|^{-d-\sigma}$, $\sigma>0$, and assume $d<4$. For $\sigma<d/2$, the critical behavior is mean-field-like. For $\sigma>d/2$, nontrivial critical behavior results, and there exists a renormalization-group epsilon expansion in powers of $\epsilon_\sigma=2\sigma-d$ [18]. Furthermore, the critical correlation decay exponent η is given *exactly to all orders* in ϵ_σ by $\eta=2-\sigma$. This should be compared with the results above for $\zeta(y)$. For what values of σ are the results for *short-ranged* interactions recovered? The naive answer is for $\sigma>2$, for then J_{ij} possesses a second moment, and its Fourier transform is $\hat{J}(k)=J_0(1+a_1k^2+a_\sigma k^\sigma+\cdots)$, i.e., the singular k^σ term is now subdominant to the usual k^2 short-ranged term. In fact this is incorrect; rather, one finds that short-ranged behavior results when $\sigma > 2-\eta_0$, where η_0 is the short-ranged value of η [18]. However, $\eta_0(d)$ is an entirely nontrivial function which cannot be inferred at any order in the ϵ_0 expansion, but must be calculated directly from the short-ranged Ising model. In critical phenomena this calculation can be done via the usual $\epsilon = 4-d$ expansion. However, short-ranged turbulence has no upper critical dimension, and therefore there exists no independent calculational scheme for determining where the crossover from long-ranged to short-ranged driving occurs.

We now see the beauty of the large-N method. In equations (8) the driving function $D(\mathbf{k},\omega)$ is *completely arbitrary* and the detailed crossover between short- and long-ranged $D(k)$ can be elucidated. Although we pay the price of having $N \to \infty$, we shall see that some very general statements can be made.

We first note that (8) reproduces the y-expansion results precisely. If the scaling forms (9) are substituted in (8) one finds that so long as $d < \Delta < d+z$ all integrals converge in the scaling limit $\nu_0 \to 0$. The scaling relation $\Delta + z = d+2$ results as a consistency condition. A second relation results when one assumes that $D(k)$ dominates the scaling, and hence that k^{4-d-y} scales in the same way as the nonlinear term in (8b). This yields $2\Delta - z = 2d-2+y$, and leads directly to the exponent values quoted below (9). This is perhaps not surprising since Galilean invariance plays a key role in determining these values, and we have seen that the $N \to \infty$ limit possesses a generalized Galilean invariance. What is perhaps more surprising is that one may perform the renormalization-group y expansion *directly* on Eqs. (8), and the result is *precisely* the $O(y)$ recursion relations from Ref. [4]. Thus Eqs. (8) are an exact integration of these lowest-order recursion relations [19].

However, Eqs. (8) also show where the renormalization-group approach breaks down. Specifically, at $y=3$ one has $z=1$, $\Delta = d+1$, and the convergence condition $\Delta < d+z$ fails [20]: The integrals in (8) diverge at small q. To proceed further one must put a small-k cutoff m_0 into $D(k)$, and look for scaling when $k \gg m_0$. One first isolates the divergent parts of (8) as $m_0 \to 0$. Amazingly enough this determines the scaling of G exactly: One finds $z=1$ and the equation $1/g(s) = -is + g(s)$, i.e.,

$$g(s) = is/2 - (1-s^2/4)^{1/2}. \tag{10}$$

The amplitudes satisfy $A_1\tilde{\nu} = 1$, and $A_1 A_2 \lambda_0 u_0 m_0^{d+1-\Delta} = 1$, where $u_0 = [(d-1)m_0^{\Delta-d-1}/dA_2\tilde{\nu}]\int_{\mathbf{k},\omega} U(\mathbf{k},\omega)$ is a cutoff-dependent measure of the total kinetic energy density, and remains finite as $m_0 \to 0$. Note that for $|s| > 2$, g is purely imaginary, while for $|s| \le 2$, $|g| = 1$. The only condition on $u(s)$ at this stage is that it *vanish* for $|s| \ge 2$; the exponent Δ is yet to be determined. An equation for $u(s)$ results by examining the *finite parts* of the integrals as $m_0 \to 0$. These may be combined [10] into a single integral equation for $u(s)$ and Δ, with the driving spectrum still appearing as an inhomogeneous term. Once again, assuming that the driving term determines the scaling, we find $\Delta = d + (y-1)/2$, and the equation for $u(s)$ reads

$$\delta_0 = \int_{\mathbf{p},t} b(\hat{\mathbf{k}},\mathbf{p})\, u\left(\frac{s-pt}{|\hat{\mathbf{k}}-\mathbf{p}|}\right) |\hat{\mathbf{k}}-\mathbf{p}|^{-\Delta} \left[u(s)\left(\frac{1-t^2/4}{1-s^2/4}\right)^{1/2} \theta(1-t^2/4) - p^{1-\Delta}u(t)\right], \tag{11}$$

$$|s| \le 2,\ \Delta = d + (y-1)/2,$$

where $\delta_0 = D_0\lambda_0 u_0^2 m_0^{3-y}\tilde{\nu}^{-3}$ is the properly scaled driving amplitude, and $\theta(x)$ is the step function. This equation has a solution so long as Δ does not permit a solution with $\delta_0 = 0$. Although we do not have a complete proof, based on existing numerics [21] and arguments based on conformal transformations [22] we believe that when $y=4$, i.e., $\Delta = d + \frac{3}{2}$, and $\zeta = \frac{3}{2}$, such a solution exists. Thus for $y > 4$ the driving no longer controls the scaling; effectively one has $\delta_0 = 0$, and ζ sticks at the value $\frac{3}{2}$. The solution to (11) with $\delta_0 = 0$ then describes true turbulence [note that one obvious such solution is $\Delta = 1$, and $u(s) = (1-s^2/4)^{1/2}$, but this violates the condition $\Delta > d+z$ upon which the derivation of (11) was based]. We are presently pursuing numerical solutions to (11) to verify the $y=4$ borderline, and to solve for the function $u(s)$—there appears to be no simple analytic form. Note that $z=1$ and $\zeta = \frac{3}{2}$ were precisely the values found originally by Kraichnan [14,23], who also arrived at a result equivalent to (10). However, his approximate solution for u was $u(s) = (1-s^2/4)^{1/2}$, which does not satisfy (11). Equation (11) appears to be a new result.

What is the connection between our analysis and those that find the Kolmogorov results $\zeta = \frac{5}{3}$ and $z = \frac{2}{3}$ for $y > 4$? The answer is that if the divegent parts of (8) are simply *dropped*, the renormalization-group results extend to $y=4$, at which point similar numerical and conformal arguments indicate that a $\delta_0 = 0$ solution exists for a *subtracted* version of Eqs. (8), in which now both $u(s)$ and $g(s)$ are nontrivial (we emphasize that the $\zeta = \frac{5}{3}$ result, though widely accepted, has never been proven for the subtracted DIA, and therefore has precisely the same status as $\zeta = \frac{3}{2}$ for the unsubtracted DIA [22]. However, since the unsubtracted Eqs. (8) are fundamental, this procedure is seen to be arbitrary and inappropriate.

What general conclusions can we make? First, the breakdown of the renormalization-group results at $z=1$ is almost certainly exact, and the value $z=1$ thereafter is probably general. This confirms the "random Taylor hypothesis," namely, that sweeping of small scales by large ones should make the spatial and temporal spectra very similar. The exponents Δ, ζ are still nontrivial, and presumably vary with N. If indeed $\zeta(N \rightarrow \infty) = \frac{3}{2}$ this is remarkably close to the experimental result $\zeta \simeq \frac{5}{3}$, differing by only 10%. Large N expansions in critical phenomena seldom do this well. Calculating the next correction in powers of $1/N$ is a daunting task [7,13], but seems to be a necessary step in order to confirm these ideas.

We end by emphasizing the philosophy of our approach. The idea we propose is that the DIA Eqs. (8) represent an exact solution in a special limit, which is *continuously related, via N, to the real turbulence problem. These equations* should thus be taken *at face value.* Previous work [20] which has concentrated on modifying them to obtain the $\frac{5}{3}$ law thus appears to miss the mark. We view the $\frac{3}{2}$ law not as a problem to be fixed, but rather as an amazingly accurate zeroth-order approximation in a systematic expansion for $\zeta(N)$. The closeness of the experimental value to the Kolmogorov $\frac{5}{3}$ is a perhaps unfortunate coincidence which has led people away from taking the DIA equations as seriously as they deserve to be taken.

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[1] See, e.g., A. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1975), Vol. 2.

[2] A. N. Kolmogorov, C.R. Acad. Sci. U.S.S.R. **30**, 301 (1941); **32**, 16 (1941).

[3] See, e.g., T. S. Lundgren, Phys. Fluids **25**, 2193 (1982).

[4] (a) D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977); (b) C. DeDominicis and P. C. Martin, Phys. Rev. A **19**, 419 (1979).

[5] R. Abe, Prog. Theor. Phys. **49**, 113 (1973); S.-K. Ma, Phys. Rev. A **7**, 2172 (1973).

[6] D. J. Amit and D. V. I. Roginsky, J. Phys. A **12**, 689 (1979).

[7] We would be more comfortable enlarging the group G_N with N, choosing the dimension of the representation as small as possible. This is more in line with the standard procedure from critical phenomena (Ref. [5]). However, even the O(3) Wigner symbols, which are by far the most carefully studied, are extremely complicated: In Ref. [6] their large-l behavior had to be studied *numerically* to ascertain the correct $N \rightarrow \infty$ limit for the Potts model. The same problem exists in the application to turbulence. Although Wigner symbols may be defined for any group, we do not, at this stage, even know if comparable technology exists for their evaluation for groups other than O(3). We are indebted to Andreas Ludwig for discussions on this point.

[8] See, e.g., G. Baym, *Lectures on Quantum Mechanics* (Benjamin/Cummings, Reading, MA, 1969), Chap. 17.

[9] For a thorough discussion of the symmetry and orthogonality properties of the $3j$ symbols, see L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977), 3rd ed., Chap. 14.

[10] Details will be published elsewhere.

[11] Galilean invariance greatly simplifies the renormalization properties of the field theory associated with Eq. (4) (see Ref. [4(b)]). We suspect that similar use may be made of the generalized invariances (8), though we have yet to check this.

[12] H. W. Wyld, Ann. Phys. (N.Y.) **14**, 143 (1961); P. C. Martin, E. D. Siggia, and H. Rose, Phys. Rev. A **8**, 423 (1973).

[13] It is tempting to speculate that the next correction in powers of $1/N$ will, just as in the critical phenomena of the O(N) models, consist of graphs with one less bubble, i.e., a single crossing line. However, not enough is known about the asymptotics of the $3j$ symbols to conclude this. The question may only be decidable numerically, and it is possible that there will be *no* natural organization of the diagrams beyond lowest order in $1/N$, i.e., that the O(3) formulation of the $1/N$ expansion does not even exist. One would then be forced to seek a more standard formulation with a sequence of groups G_N, growing with N (Ref. [7]). These are difficult questions, but crucial, and must be addressed in the future.

[14] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).

[15] Kraichnan has also noted that the DIA equations result from an $N \rightarrow \infty$ limit, in which the $A_{\mathbf{k}}^{\alpha\beta\gamma}$ in (4) are *randomly* $\pm 1/N$, restricted only by symmetry under permutation of the indices. See R. H. Kraichnan and S. Chen, Physica (Amsterdam) **37D**, 160 (1989), for a review. We stress, however, that only when the generalized fluid equations include some higher-order symmetry varying with N do we expect the universal exponents to vary *continuously* with N, thereby yielding a systematic expansion. We do not expect a finite set of randomly coupled velocity fields to respect this requirement. Kraichnan's result does, however, lead one to expect that the limit $N \rightarrow \infty$ will be rather insensitive to the detailed procedure for obtaining it (Ref. [7]).

[16] See, e.g., V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1986).

[17] This analogy seems to originate from an oral presentation by P. C. Hohenberg, and was brought to our attention by M. C. Cross (private communication).

[18] See J. Sak, Phys. Rev. B **8**, 281 (1973), and references therein.

[19] It is actually easy to see why this must be so: the order y recursion relations are obtained precisely by renormalizing the same bubble diagrams that survive at $N \rightarrow \infty$ (see Ref. [4(a)]). The renormalization-group recursion relations therefore result essentially from applying the method of characteristics to the DIA equations. This is apparently very well known to those who know it well [E. D. Siggia (private communication)], but we are aware of only partial results in this direction [see J. Bhattacharjee, J. Phys. A **21**, L551 (1988)].

[20] This boundary is well known, and comes from the oft-quoted "sweeping of small eddies by large ones." Much effort has been expended trying to remove it [see, e.g., R. H. Kraichnan, Phys. Fluids **8**, 575 (1965); V. L'vov, Phys. Rep. **207**, 1 (1991)].

[21] R. H. Kraichnan, Phys. Fluids **7**, 1030 (1964); J. Herring, Phys. Fluids **9**, 2106 (1966).

[22] E. A. Kuznetsov and V. S. L'vov, Physica (Amsterdam) **2D**, 203 (1981).

[23] See also S. A. Orszag, in *Fluid Dynamics,* edited by R. Balian and J. L. Peube (Gordon/Breach, London, 1977).