Description of Chaos-Order Transition with Random Matrices within the Maximum Entropy Principle

M. S. Hussein and M. P. Pato
Nuclear Theory and Elementary Particle Phenomenology Group, Instituto de Física Universidade de São Paulo, CP 20.516, 01498-970, São Paulo, SP, Brazil
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The deformed Gaussian orthogonal ensemble introduced earlier is developed here for large dimensional matrices. Both the spacing and eigenvector distributions are studied and compared to other ones suggested for the chaos-order transition problem. The concept of a universal lower entropy with respect to the Gaussian orthogonal ensemble entropy is proved very useful.

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It is expected that for systems whose classical motion is neither regular nor fully chaotic, the statistical behavior is intermediate between the Poisson and the Gaussian orthogonal ensemble (GOE) limits [1-16]. Several authors have suggested empirical functional forms for the level spacing distributions. We cite here the Brody distribution [11], the Berry-Robnik distribution [5], and the Robnik distribution [6]. What one is usually seeking are intermediate distributions that exhibit a degree of universality close to that of their Poisson and Wigner (GOE) limits [17]. Further, the distribution of eigenvectors of a system that is fully chaotic (GOE) is known to be of the Porter-Thomas form [18]. It is therefore hoped that in the intermediate case the spacing distribution alluded to above dictates to some extent the form of the eigenvector distribution. This way one would have a fully universal description of systems intermediate between chaos and order.

Several of the above questions have been discussed in the past. In particular we cite the work of Alhassid and co-workers [19-22], Lenz and Haake [7,8], and Guhr and Weidenmüller [23]. Our aim in the present work is to develop a general framework through which all of the above questions can be addressed. We shall show that it is possible to derive a joint distribution for the spacings and eigenvectors valid in the intermediate regime. From this distribution, the spacing distribution is obtained by integrating out the eigenvectors and similarly for the eigenvector distribution. The cases of 2×2 and 3×3 matrices have already been worked out analytically [24,25]. Here we present a thorough numerical study for large dimensional matrices.

Our theory is based on the maximum entropy principle, which we briefly outline in the following.

We define the entropy associated with the distribution $P(H)$ of the Hamiltonian ensemble H as

$$S=-\int dH\,P(H)\ln P(H)\,. \tag{1}$$

We now maximize S subject to the usual constraints of the GOE,

$$\langle \mathrm{Tr}H^2\rangle \equiv \int dH\,P(H)\,\mathrm{Tr}H^2=\mu\,, \tag{2}$$

$$\langle 1\rangle =1\,, \tag{3}$$

and obtain

$$P_{\mathrm{GOE}}(H)=\exp[-\lambda_0-1-\alpha_0 \mathrm{Tr}H^2]\,, \tag{4}$$

$$\alpha_0=N(N+1)/4\mu\,,\qquad \exp(-\lambda_0-1)=2^{-N/2}(\pi/2\alpha_0)^{-N(N+1)/4}\,. \tag{5}$$

Denoting the eigenvalues by $E_1,E_2,\ldots,E_N$ and amplitudes by $C_1,C_2,\ldots,C_N$, one can easily obtain the joint distribution function

$$P(E_1,E_2,\ldots,E_N;C_1,C_2,\ldots,C_N)\equiv P(E_1,\ldots,E_N)P(C_1,\ldots,C_N) \tag{6}$$

from which the spacing distribution $P(s)$ and amplitude (eigenvector) distribution $P(c)$ can be derived. When $N\to\infty$, we obtain the Wigner distribution for $P(s)$ and the Porter-Thomas distribution for $P(c)$.

We should emphasize that the joint distribution function Eq. (6) implies no correlations between the E and C distributions. This is a consequence of the GOE, namely, $P(H)$ is invariant under arbitrary rotation of the basis.

Before turning our attention to the intermediate case, it is helpful to mention that constraint (2) gives rise to the Gaussian distribution (4) with the second moment

$$\langle H_{ij}H_{kl}\rangle=(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{jk})/4\alpha_0\,, \tag{7}$$

with α_0 independent of the label. Further, the GOE entropy can be straightforwardly derived from (1), (4), and (5). We find for the entropy per degree of freedom [there are $N(N+1)/2$ degrees of freedom for our symmetric real matrices]

$$s_{\mathrm{GOE}}=\frac{1}{2}\left(1+\ln\frac{\pi}{2\alpha_0}\right)+(N+1)^{-1}\ln 2\,. \tag{8}$$

Thus simple universal features of the intermediate distribution we are seeking are (1) a second moment that depends on the label, and (2) an entropy per degree of freedom that is *smaller* than s_{GOE}, Eq. (8).

Within the maximum entropy principle, an intermediate distribution can be defined through the addition of more constraints. Here we use the simplest possible one that allows α_0 to depend on the label. If we divide the random matrix H into four blocks and introduce the following notation (see Fig. 1),

$$H = PHP + QHQ + PHQ + QHP\,, \qquad P \equiv \sum_{i=1}^{M} |i\rangle\langle i|\,, \tag{9}$$

with $P+Q=1$, $P^2=P$, $Q^2=Q$, $PQ=QP=0$, then the desired constraint reads

$$\langle \mathrm{Tr}(PHQHP)\rangle = \nu\,. \tag{10}$$

We now maximize S subject to the GOE constraints (2) and (3) and the new one (10), to obtain the intermediate distribution. By fixing the value of $\langle \mathrm{Tr}(PHQHP)\rangle$, with a Lagrange multiplier β, we are deforming the GOE. Of course the system still maintains full axial symmetry about the P "direction." The new ensemble, which we called the deformed Gaussian orthogonal ensemble (DGOE) in Ref. [24], is invariant under a transformation that leaves vectors in P unchanged. Further understanding of the ensemble can be gained by spelling out the second moment,

$$\langle H_{ij}H_{kl}\rangle = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{1}{4\alpha + 2\beta|\delta_{ki} - \delta_{li}|}\,, \tag{11}$$

clearly showing the label dependence mentioned above.

The DGOE distributions we obtain have the general form [24]

$$P_{\mathrm{DGOE}}(H) = P_{\mathrm{GOE}}(H)\exp[-\beta\,\mathrm{Tr}(PHQHP)] \times [1+\beta/2\alpha]^{M(N-M)/2}\,, \tag{12}$$

$$\mu = \frac{N(N+1)}{4\alpha} - 2\nu\frac{\beta}{2\alpha}\,,\quad \nu = \frac{M(N-M)}{4\alpha(1+\beta/2\alpha)}\,.$$

The information content I of the DGOE relative to the GOE [26,27] is easily obtained,

$$I \equiv s_{\mathrm{GOE}} - s_{\mathrm{DGOE}} = \frac{M(N-M)}{N(N+1)}\ln\left(1+\frac{\beta}{2\alpha}\right)\,. \tag{13}$$

Equation (13) clearly shows that a system described by the DGOE is less chaotic, since the difference $s_{\mathrm{DGOE}} - s_{\mathrm{GOE}} < 0$. Further, the degree of order in the DGOE is measured by both M, the dimension of the symmetric nondiagonal block matrix, and β. For very large matrices ($N \to \infty$), the DGOE is not much different from the GOE if M is taken to be small. For M comparable to N, namely, $N \to \infty \Rightarrow M/N \equiv n < 1$ finite, we obtain a saturation limit for I,

$$I = n(1-n)\ln(1+\beta/2\alpha)\,. \tag{14}$$

Our detailed numerical calculation described below corroborates our discussion above, namely, that for a fixed value of β/α, the amplitude and the spacing distributions saturate with respect to the dimension of the matrix. This is the universal feature we are seeking.

Before presenting the numerical results we mention that in Ref. [24] we have worked out fully analytically the cases of 2×2 and 3×3 matrices. The conclusion reached by us as well as by Refs. [6] and [20] is that level repulsion goes away when β is rigorously set equal to ∞. This, when generalized to larger matrices, indicates that the $\beta \to \infty$ limit must correspond to two *decoupled* GOE's and $\beta \to 0$ to the case of two fully mixed GOE's (and thus a single doubly larger GOE). In fact it can be shown that our DGOE can be reformulated in such a way that the quantity $1/(1+\beta/2\alpha)^{1/2} \equiv \lambda$ acts as a coupling constant in a description involving the following Hamiltonian:

$$H(\lambda) = (PH_GP + QH_GP) + \lambda(PH_GQ + QH_GP)\,, \tag{15}$$

$$\equiv H_0 + \lambda V\,, \tag{16}$$

λ taking the value from 0 ($\beta = \infty$), which is the regular case, to 1 ($\beta = 0$), which is the fully chaotic case [$H(\lambda = 1) = H_G$]. Note that PH_GP, QH_GQ, PH_GQ, and QH_GP are all random matrices.

Several authors have addressed the problem of chaos-order transition using the decomposition (16) for H. In particular we mention Guhr and Weidenmüller [23], who treat the problem of isospin mixing in compound nuclear reactions. There is also the work of Lenz and Haake [7,8] who consider a more general case of H_0 and V belonging to different ensembles (e.g., H_0: GOE; V: Gaussian unitary ensemble). Alhassid and Levine [19] considered the same problem using Dyson's random walk formulation and they obtained the s and c distributions for the 2×2 matrix case. The way we formulate the chaos-order transition, through the DGOE, Eq. (12), allows a realistic large dimensional numerical study of *both* the spacing and the eigenvector distributions. Before proceeding we mention that for large matrix Hamiltonians H_0 and V, the DGOE information content I is given by [see Eq. (13)]

$$I \equiv s_{H_G} - s_H = 2n(1-n)\ln(1/\lambda)\,. \tag{17}$$

For $\lambda = 0$, I is infinitely positive. Equation (17) is an interesting way to quantitatively measure how much more information is contained in $H(\lambda)$ with respect to H_G.

We have considered an ensemble of matrices of varying

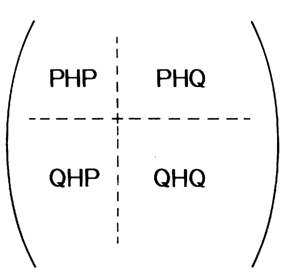

FIG. 1. The block structure of the Hamiltonian. See text for details.

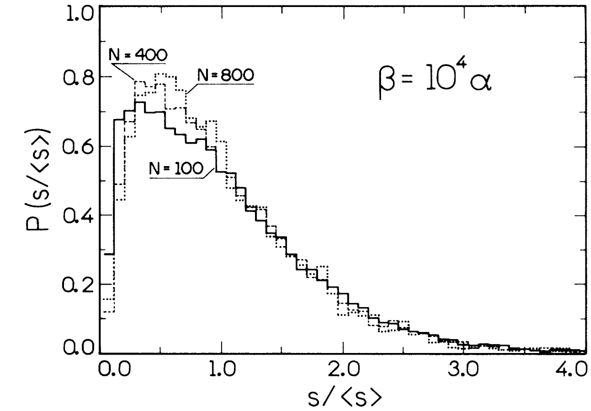

FIG. 2. The spacing distribution of the DGOE with $\beta=10^4\alpha$ for different dimensions of *PHP* ($\frac{1}{2}N$).

dimensions pertaining to the DGOE as described before. The spacing distribution for $\beta=10^4\alpha$ is shown in Fig. 2 for $N=100$, 400, and 800. We took $M/N=1/2$. We see clearly that as N increases the distribution becomes independent of N as our entropy argument based on Eq. (14) tells us. We next turn to the variation of the distribution for large enough N ($N=800$) with β. This is shown in Fig. 3. Also shown in the figure are the corresponding Dyson-Mehta Δ_3 distributions [28]. The shift of the maximum from the Wigner one towards small s is gradual. At very large values of β (small λ), the distribution is not exactly Poissonian simply because, though repulsion is gone, there is still level repulsion in the *PHP* and *QHQ* block matrix distributions. In this respect, we are in complete agreement with Guhr and Weidenmüller [23]. Note that we have verified that the Δ_3 behavior saturates at large β below the Poisson value. Note also that the parameter α of Ref. [23] corresponds to λ of Eq. (16) and thus is equal to $1/(1+\beta/2\alpha)^{1/2}$.

We next turn to the eigenvector distribution. We plot in Fig. 3(c) $P(y)$ vs $\ln(y/\langle y\rangle)$; $y\equiv c^2$. We show the histogram of our numerical diagonalization, the Porter-Thomas distribution (dashed curve), and the χ^2 distribution suggested by Alhassid and Levine [19],

$$P_\nu(y)=\left[\frac{\nu}{2\langle y\rangle}\right]^{\nu/2}\frac{y^{\nu/2-1}\exp[-\nu y/2\langle y\rangle]}{\Gamma(\nu/2)}\,,\qquad y\equiv c^2\,, \tag{18}$$

where ν is a parameter that measures the number of degrees of freedom. When $\nu=1$, one recovers the *PT* distribution. From the figure, we see clearly that when constraining both the spacing and the amplitude distribution through the DGOE, the resulting $P(y)$ deviates appreciably from Eq. (18), for large values of β (small λ).

The value of ν that best fits the "data" is determined by inverting the equation [19]

$$\langle \ln y/\langle y\rangle\rangle=\psi(\nu/2)-\ln(\nu/2)\,, \tag{19}$$

where $\psi(x)$ is the digamma function. The quantity $\langle \ln y\rangle$ which is constructed from the data is plotted in Fig. 4 for several values of N ($M=N/2$) vs n/N, where n denotes

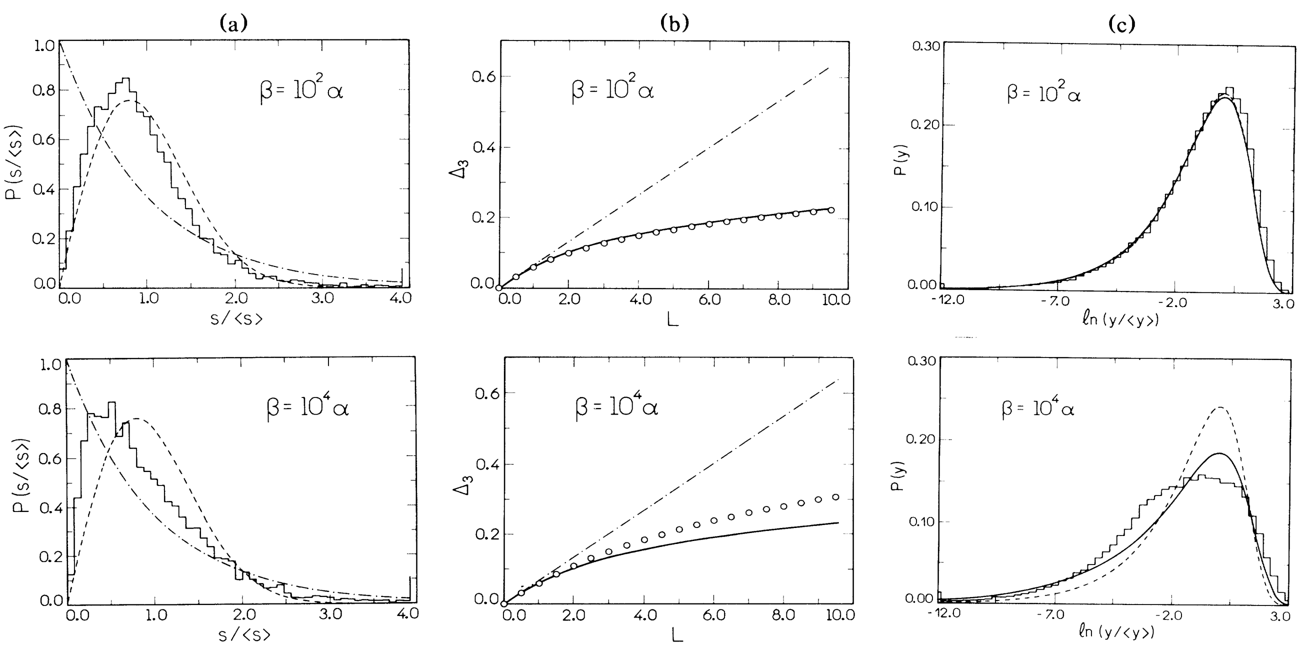

FIG. 3. (a) The spacing distribution $P(s)$, (b) the spectral rigidity $\Delta_3(L)$, and (c) the intensity distribution $P(y)$, for two values of β. The dimension of the Hamiltonian matrix is 700 and that of *PHP* is 350.

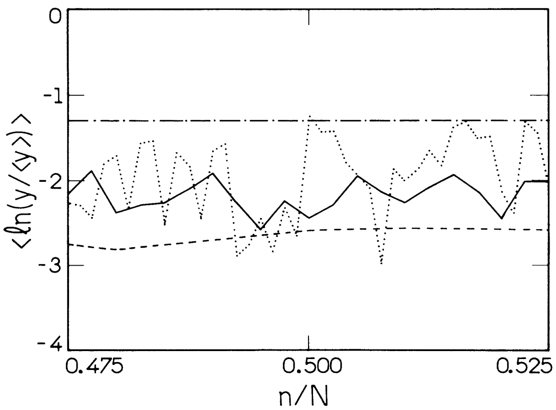

FIG. 4. The surprisal, $\langle \ln y/\langle y\rangle\rangle$, vs n/N, where n represents the label of the eigenvector coefficient (see text for details). Dashed curve, $N=100$; full curve, $N=400$; and dotted curve, $N=800$. The dash-dotted curve represents the Porter-Thomas result.

the label of the eigenvector coefficient $|E_k\rangle=\sum_n C_n^k|n\rangle$. We see a great amount of fluctuation that is smoothed out when averaging over an ensemble of matrices (the curve for $N=400$). We verified that the ensemble average is close to the n average. The larger dimensional cases ($N=600$ and 800) are seen to fluctuate a lot and their average, needed to obtain ν above, was found by the simple n average mentioned above. We notice from the figure a certain degree of saturation is attained for $N=600$. The average for $N=600$ is close to that for $N=800$.

We should mention that the amplitude distribution for very large β shown in Fig. 3(c) which deviates appreciably from the χ^2 distribution, Eq. (18), corresponds to a situation of two almost completely decoupled GOE's (PHP and QHQ), since $\lambda \sim 0.0$. For such a case we verified that a better account of the data (histogram) is obtained by using a sum of two χ^2 distributions, one, $aP_\nu(y)$, for PHP and the other, $bP_\mu(y)$, for QHQ. The surprisal procedure used by Alhassid and Levine can be applied for this more general case to find a, b, ν, and μ.

In conclusion, we have presented a detailed numerical study of the deformed Gaussian orthogonal ensemble [24] for large matrices. Our guiding principle has been the dimension independence of the reduction in the entropy with respect to that of the GOE. The general conclusion drawn from our study is that it is possible to discuss both the spacing distribution and the amplitude distribution using the same ensemble appropriate for the intermediate situation between chaos and order. Our theory should be useful for the study of nuclear statistics and symmetry breaking, as well as the general question of chaos-order transition. We are presently applying our theory to the isospin mixing problem considered in Ref. [23]. We end by stating that the full Poisson limit (no level repulsion) is attained within the DGOE by considering H to be block diagonal with the block matrices having 2×2 dimensions.

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