Lineal Gravity from Planar Gravity

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The 2D "black hole" action is obtained by a *nonstandard* dimensional reduction of pure 3D gravity with a nonzero cosmological constant. Starting from the Chern-Simons formulation of 2+1 gravity, we obtain the (1+1)-dimensional gauge formulation given by Verlinde. Remarkably, the proposed reduction shares the relevant features of the formulation of Cangemi and Jackiw, without the need for a central charge in the algebra. The Lagrange multipliers in these formulations appear naturally as remnants of the three-dimensional theory. The proposed dimensional reduction involves a shift in the three-dimensional connection whose effect is to make the physical length of the extra dimension infinite.

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The past few months have seen a revival in the study of the quantum properties of black holes due to the discovery that relatively simple actions in two dimensions admit black hole solutions [1,2]. These are being intensively studied as toy models in which to investigate backreaction in Hawking radiation and related issues in quantum gravity, in particular the formation of black holes and the final stages of black hole evaporation [2,3]. The hope is to find a consistent quantization scheme for 2D gravity coupled to matter that will answer some of the questions that are intractable in four dimensions.

We expect the underlying group structure to be crucial in the quantization process. Exact quantization of three-dimensional gravity was achieved by Witten using the fact that it can be rewritten as a Chern-Simons action for the tangent space group [4,5]. The two-dimensional "Einstein" action is a topological invariant and therefore has trivial dynamics. Alternatives include the action proposed by Teitelboim [6] and Jackiw [7] and the above-mentioned "string-inspired" actions [2,8]. The gauge-theoretical formulation of the former [6,7], based on the group SO(2,2), was obtained by Chamseddine and Wyler [9], Isler and Trugenberger [10], and Fukuyama and Kamimura [11]. The group formulation of the string-inspired action has recently been found in two very interesting papers, which offer different answers. The formulation proposed by Verlinde [8] is loosely based on the Poincaré group ISO (1,1). That of Cangemi and Jackiw [12] is, in some sense, more "natural," but at the expense of introducing a central charge in the algebra.

A general feature of all these two-dimensional actions is that their gauge-theoretical formulations include extra fields (Lagrange multipliers) which do not come from the metric, and whose geometric interpretation is unclear. In this paper we find a geometric interpretation of these Lagrange multipliers by a process of dimensional reduction from pure gravity in three dimensions. They are remnants of the three-dimensional theory associated with the generators that disappear in the reduction. In the process, we recover Verlinde's formulation of the black hole action, but in a way that shares the good features of the formulation of Cangemi and Jackiw. Our starting point is pure (2+1)-dimensional gravity (with a cosmological constant) in its Chern-Simons formulation. For completeness, and because it illustrates the main features of the reduction very clearly, we first obtain the action of [6,7] by a perfectly standard dimensional reduction. We then propose a nonstandard reduction scheme involving a shift in the three-dimensional connection, which yields the black hole action of [2,8].

We begin by considering the Einstein action for (2+1)-dimensional gravity with a cosmological constant

$$S=\int d^3x\sqrt{g}\,(R_g-2\Lambda)\,, \tag{1}$$

where g is the determinant of the metric $g_{IJ}(I,J=0,1,2)$, and R_g is the Ricci scalar. $(-\Lambda)$ is the cosmological constant; in what follows we shall take it to be negative, i.e., $\Lambda>0$, but the same analysis goes through for $\Lambda<0$. Our conventions are as follows: We will use capital letters for three-dimensional indices and lower case letters for two-dimensional ones. $I,J,i,j\ldots$ are space-time indices, while $A,B,a,b\ldots$ are tangent space indices. The former are raised and lowered with the three-dimensional metric g_{IJ} or the two-dimensional metric γ_{ij}, the latter with the Minkowski metrics $\eta_{AB}=(+1,-1,-1)$ or $\eta_{ab}=(+1,-1)$. ϵ_{ABC} is the totally antisymmetric symbol with $\epsilon^{012}=1$. In two dimensions, $\epsilon^{ab}=\epsilon^{ab2}$, so $\epsilon^{01}=1$ also.

One can also formulate the action (1) in a first-order formalism. Given coordinates (x^I), one introduces an orthonormal triad $\{\mathbf{e}_A\}$, with components $\mathbf{e}_A=e_A{}^I\partial/\partial x^I$ in the coordinate basis, and a spin connection $\{\omega_I{}^A\}$ such that the torsion constraints $\epsilon^{IJK}(\partial_J e_K{}^A-\epsilon^A{}_{BC}\omega_J{}^B e_K{}^C)=0$ are satisfied. If the $e_I{}^A$ are the inverses of the $e_A{}^I$, then the metric can be written $g_{ij}=e_I{}^A e_J{}^B\eta_{AB}$. The three-dimensional Riemann tensor is now

$$R_{IJKL}=\epsilon_{ABC}e_K{}^A e_L{}^B[\partial_I\omega_J{}^C-\tfrac{1}{2}\epsilon^C{}_{DF}\omega_I{}^D\omega_J{}^F]-(I\rightarrow J)$$

and the equation $R-2\Lambda=0$ becomes

$$\partial_I\omega_J{}^A-\tfrac{1}{2}\epsilon^A{}_{BC}\omega_I{}^B\omega_J{}^C-\tfrac{1}{2}\Lambda\epsilon^A{}_{BC}e_I{}^Be_J{}^C-(I\to J)=0\,.$$

The second-order formulation is recovered by solving the torsion constraints for the ω's as a function of the e's and substituting $\omega(e)$ in the last equation.

We can rewrite the equations in a coordinate free way by introducing the one-forms e^A and ω^A (here, $e^A=dx^Ie_I{}^A\omega^A=dx^I\omega_I{}^A$). We then have

$$\begin{aligned}T^A&=De^A=de^A-\epsilon^A{}_{BC}\omega^Be^C=0\,,\\R^A&=d\omega^A-\tfrac{1}{2}\epsilon^A{}_{BC}\omega^B\omega^C-\tfrac{1}{2}\Lambda\epsilon^A{}_{BC}e^Be^C=0\,.\end{aligned}\tag{2}$$

If we introduce an SO(2,2) connection $A=e^AP_A+\omega^AJ_A$ where P_A and J_A are translations and Lorentz rotations, satisfying

$$\begin{aligned}&[P_A,P_B]=-\Lambda\epsilon_{ABC}J^C,\quad[J_A,P_B]=-\epsilon_{ABC}P^C\,,\\&[J_A,J_B]=-\epsilon_{ABC}J^C\,,\end{aligned}\tag{3}$$

the equations (2) are immediately recognized as the components of the curvature two-form of SO(2,2) $F=dA+\frac{1}{2}[A,A]=T^AP_A+R^AJ_A$ and the first-order action corresponding to (1) turns out to be the Chern-Simons action for this connection [4,5],

$$S=\int\mathrm{tr}(AdA+\tfrac{2}{3}A^3)\,,\tag{4}$$

where the exterior product of differential forms is understood. [The same is true for SO(3,1) if the cosmological constant is positive, or for ISO(2,1) if it is zero.]

The "trace" in the action refers to any invariant bilinear form in the algebra, and in this case it is given by the Casimir $C=P_AJ_B\eta^{AB}$. Note that the algebra is semisimple, so(2,2) =so(2,1)+so(2,1), with the two factors generated by $M_A^\pm=\frac{1}{2}(J_A\pm P_A/\sqrt{\Lambda})$. This means that there is a one-parameter family of Casimirs (up to overall normalization) $C(\sigma)=M_A^+M^{+A}+\sigma M_A^-M^{-A}$. All the actions obtained for different values of σ are classically equivalent in the sense that their equations of motion are the same, $F^+=F^-=0$. The Einstein action is obtained for $\sigma=-1$.

Another important point is that the Chern-Simons formulation trades diffeomorphism invariance for gauge invariance. Both symmetries differ by a third, "trivial," symmetry where the variation of one field is proportional to the equation of motion of another, in such a way that the total variation of the action cancels (*off shell*) [4,5].

From now on, the indices a,b will take the values 0,1 only, and we will indicate the index 2 explicitly.

We begin with the action proposed by Teitelboim [6] and Jackiw [7]

$$S=\int d^2x\sqrt{-\gamma}\Phi(R_\gamma-2\Lambda)\tag{5}$$

($\gamma=\det\gamma_{ij}$). Φ is a Lagrange multiplier enforcing the constraint $R_\gamma-2\Lambda=0$. In order to obtain this action in the second-order formulation by dimensional reduction of the three-dimensional Einstein action, we consider coordinates $(x^I)=(x^i,y)$, we compactify the y coordinate by identifying the points whose y coordinate differs by L, and we impose that all derivatives of the metric in the y direction be zero. In addition, we parametrize the three-dimensional metric as

$$g_{IJ}\begin{bmatrix}\gamma_{ij}-A_iA_j & \Phi A_i\\ \Phi A_j & -\Phi^2\end{bmatrix},\quad\partial_y(g_{IJ})=0\,.\tag{6}$$

For simplicity we will impose the ansatz $A_i=0$ (we show later that our results hold even with nonzero A_i's). Then, $\sqrt{g}=\Phi\sqrt{-\gamma}$, and (1) yields (5) plus a total derivative. This is the dimensional reduction that was proposed in [7].

Dimensional reduction is also straightforward in the first-order formulation. Starting with (4) one obtains the gauge-theoretical formulation of (5) given in [9–11]. They introduce a gauge connection $A=e^aP_a+\omega J$ for the 2D anti-de Sitter group SO(1,2),

$$[P_a,P_b]=\Lambda\epsilon_{ab}J,\quad[P_a,J]=\epsilon_{ab}P^b\tag{7}$$

and a triplet of Lagrange multipliers χ_A transforming under the coadjoint representation. The action is written

$$S=\int\chi_AF^A=\int[\chi_a(de^a+\epsilon^{ab}\omega e_b)+\chi_2(d\omega+\tfrac{1}{2}\Lambda\epsilon_{ab}e^ae^b)]\,,\tag{8}$$

where F^A are the components of the SO(1,2) curvature,

$$\begin{aligned}F&=dA+\tfrac{1}{2}[A,A]\\&=(de^a+\epsilon^{ab}\omega e_b)P_a+(d\omega+\tfrac{1}{2}\Lambda\epsilon_{ab}e^ae^b)J\,.\end{aligned}\tag{9}$$

To obtain this formulation by dimensional reduction, we need an ansatz for the e^A,ω^A equivalent to the previous ansatz on the metric. Since $g_{IJ}=e_I{}^ae_J{}^b\eta_{ab}-e_I{}^2e_J{}^2$, the dreibein components are independent of y:

$$e_y{}^2=\Phi(x),\quad e_i{}^2=A_i(x),\quad e_i{}^ae_j{}^b\eta_{ab}=\gamma_{ij}(x)\,.\tag{10}$$

Setting $A_i=0$, the connection one-forms become

$$e^a=dx^ie_i{}^a,\quad e^2=dy\,\Phi\,.\tag{11}$$

We now look at the ansatz for the spin connection. Any ansatz used in the dimensional reduction must be consistent with the equations of motion of the three-dimensional theory. Introducing $\omega^A=dx^i\omega_i{}^A+dy\,\omega_y{}^A$ in the Eqs. (2), we find that consistency requires $\omega_i{}^a=\omega_y{}^2=0$, and therefore the correct ansatz is

$$\omega^a=dy\,\omega_y{}^a(x),\quad\omega^2=dx^i\omega_i{}^2\,.\tag{12}$$

We now perform a Wigner-Inönu contraction by rescaling the algebra generators

$$J_a \to J_a/\mu, \quad P_2 \to P_2/\mu . \qquad (13)$$

Note that J_2 cannot be rescaled if it is to generate 2D Lorentz rotations, since a rescaling would alter its commutation relation with translations. The rescaling of P_2 is then determined by the condition that the Casimir be nondegenerate after the contraction. The commutation relations (3) become

$$[P_a,P_2]=\Lambda\epsilon_{ab}J^b,\quad [J_a,P_2]=\mu^2\epsilon_{ab}P^b,\quad [J_a,J_b]=\mu^2\epsilon_{ab}J_2 ,$$
$$[J_a,P_b]=\epsilon_{ab}P_2,\quad [J_a,J_2]=\epsilon_{ab}P^b , \qquad (14)$$

together with those of the two-dimensional anti–de Sitter group SO(2,1),

$$[P_a,P_b]=\Lambda\epsilon_{ab}J_2,\quad [J_2,P_a]=-\epsilon_{ab}P^b . \qquad (15)$$

From now on, $J_2=J$, the two-dimensional Lorentz generator, and $\omega^2=\omega$. The rescaled Casimir

$$\mu C=P_aJ^a-P_2J \qquad (16)$$

survives in the limit $\mu\to 0$, and the action (4) becomes

$$S=\int dy\int d^2x[\omega_{ya}(de^a+\epsilon^{ab}\omega e_b) -\Phi(d\omega+\tfrac{1}{2}\Lambda\epsilon_{ab}e^ae^b)] . \qquad (17)$$

Integration over y yields the action (8) (times L). Notice that the Lagrange multipliers $\chi^a=\omega_y{}^a$, $\chi^2=e_y{}^2=\Phi$ appear naturally; they are simply the components of the three-dimensional connection corresponding to the generators J_0, J_1, and P_2 that have been lost in the two-dimensional theory.

We now turn to the string-inspired model of [2], or rather to that of [8],

$$S_2=\int d^2x\sqrt{-\gamma}(\Phi R_\gamma-\lambda) , \qquad (18)$$

which is obtained from the action of [2] by a conformal transformation.

There are two ways of writing this action in a gauge-theoretical framework. The first one, proposed by Verlinde [8] uses the ISO(1,1) group

$$[P_a,P_b]=0,\quad [P_a,J]=\epsilon_{ab}P^b \qquad (19)$$

and writes the action as ($A=0,1,2$)

$$S_2=\int(\chi_AF^A-\lambda\epsilon_{ab}e^ae^b)$$
$$=\int[\chi_a(de^a+\epsilon^{ab}\omega e_b)+\chi_2d\omega-\tfrac{1}{2}\lambda\epsilon_{ab}e^ae^b] , \qquad (20)$$

which is invariant under the transformations

$$\delta e^a=d\theta^a+\epsilon^{ab}\omega\theta_b+\epsilon^{ab}\alpha e_b,\quad \delta\chi^a=\epsilon^{ab}\alpha\chi_b+\lambda\epsilon^{ab}\theta_b ,$$
$$\delta\omega=d\alpha,\quad \delta\chi^2=\epsilon_{ab}\chi^a\theta^b . \qquad (21)$$

The second one, proposed by Cangemi and Jackiw, uses a central extension of the ISO(1,1) algebra

$$[P_a,P_b]=\lambda\epsilon_{ab}I,\quad [P_a,J]=\epsilon_{ab}P^b ,$$
$$[I,P_a]=[I,J]=0 , \qquad (22)$$

which has a nondegenerate Casimir $C=P_aP^a-\lambda IJ$. Introducing $A=e^aP_a+\omega J+\lambda aI$, the action becomes

$$S_2=\int\chi_AF^A=\int\chi_a(de^a+\epsilon^{ab}\omega e_b) +\chi_2d\omega+\chi_3(da+\tfrac{1}{2}\epsilon_{ab}e^ae^b) \qquad (23)$$

(where now A runs from 0 to 3) and is invariant under the natural gauge transformations $\delta A=dA+[A,v]$ for a gauge parameter $v=v^aP_a+vJ+\lambda uI$. Note that, in this case, $F=(de^a+\epsilon^{ab}\omega e_b)P_a+d\omega J+\lambda(da+\tfrac{1}{2}\epsilon_{ab}e^ae^b)$ and there are *four* Lagrange multipliers. Verlinde's formulation is recovered after elimination of a and χ_3 by their equations of motion, notably $d\chi_3=0$, which allows us to set $\chi_3=-\lambda$.

To obtain a 2D formulation starting with the 3D Chern-Simons action, we make the rescalings (13) *together with the constant shift*

$$e^2\to e^2+(\lambda/\Lambda)dy \qquad (24)$$

and take the limit where *both* μ *and* Λ go to zero. This procedure gives the action (18) provided $\mu^2/\Lambda\to 0$.

Recall that in both reductions (with and without shift) we imposed the ansatz $A_i=0$ on the 3D metric. It turns out that A_i terms make absolutely no difference: The extra terms one obtains in the action are multiplied by ω_y^2, which is forced to be zero by the equations of motion, and therefore the A_i's decouple from the theory.

Note that in the limit $\mu=\Lambda=0$, the algebra (14),(15) is invariant under the interchange of J_a and P_a. As a consequence, the rescaled Casimir (16) is essentially equivalent to that appearing in the ISO (1,1) algebra with central charge, proposed by Cangemi and Jackiw (even though P_2 is *not* a central charge). The first term behaves like P_aP^a and the second term plays the role of the JI term in [12], making the Casimir nondegenerate. Note that there is no rescaling that will give the central charge in the algebra while keeping J as the generator of Lorentz rotations.

With this ansatz, the Chern-Simons action becomes

$$S=\int dy\int[\omega_{ya}(de^a+\epsilon^{ab}\omega e_b)-\Phi d\omega-\tfrac{1}{2}\lambda\epsilon_{ab}e^ae^b] +\cdots , \qquad (25)$$

where $\cdots$ indicates terms that vanish in the limit $\mu,\Lambda\to 0$. This is precisely (20) and again we recognize the Lagrange multipliers as coming from ω^a and e^2.

The gauge transformations $\delta A=dv+[A,v]$ with

$$A=e^aP_a+\omega J+\omega^aJ_a+e^2P_2 ,$$
$$v=\theta^aP_a+\alpha J+\beta^aJ_a+\rho P_2 \qquad (26)$$

become

$$\begin{aligned}\delta e^2 &= d\theta^a + \epsilon^{ab}\omega^2\theta_b + \epsilon^{ab}\alpha e_b \\ &\quad + \mu^2\epsilon^{ab}[e^2 + (\lambda/\Lambda)dy]\beta_b + \mu^2\epsilon^{ab}\rho\omega_b\,, \\ \delta\omega^2 &= d\alpha + \mu^2\epsilon_{ab}\omega^a\beta^b + \Lambda\epsilon_{ab}e^a\theta^b\,, \\ \delta\omega^a &= d\beta^a + \epsilon^{ab}\alpha\omega_b + \epsilon^{ab}\omega^2\beta_b \\ &\quad + \Lambda\epsilon^{ab}\rho e_b + \Lambda[e^2 + (\lambda/\Lambda)dy]\epsilon^{ab}\theta_b\,, \\ \delta e^2 &= d\rho + \epsilon_{ab}\omega^a\theta^b + \epsilon_{ab}e^a\beta^b\,.\end{aligned} \tag{27}$$

In the limit when μ and Λ tend to zero, we recover the transformations proposed by Verlinde by setting $\beta^a = \rho = 0$. The remaining symmetries of the action change it by a total derivative, and only affect the Lagrange multipliers. They are given by

$$\begin{aligned}&\delta_\beta\omega^a = d\beta^a + \epsilon^{ab}\omega\beta_b, \quad \delta_\rho\omega^a = 0\,, \\ &\delta_\beta e^2 = -\epsilon_{ab}e^a\beta^b, \quad \delta_\rho e^2 = d\rho\,.\end{aligned} \tag{28}$$

Note that the shift $e^2 \to e^2 + (\lambda/\Lambda)dy$ corresponds to a shift $\Phi \to (\Phi + \lambda/\Lambda)$ and it has the effect of putting all points in the y direction an infinite distance away. This shift in Φ can be made directly in the action (1), or in its dimensional reduction (5)

$$S_2 = \lim_{\Lambda\to 0}\int d^2x\sqrt{-\gamma}\left(\Phi + \frac{\lambda}{\Lambda}\right)(R_\gamma - 2\Lambda)\,. \tag{29}$$

(Incidentally, this clearly illustrates that in S_1 the two-dimensional cosmological constant is the same as that of the three-dimensional theory, whereas this is not so in the string-inspired action S_2.) Note that this procedure would give rise to divergent terms in the limit where Λ goes to zero *in any dimension other than two.* The reason it is acceptable in two dimensions is that the $1/\Lambda$ term in the action happens to be multiplying the Euler characteristic, which is a topological invariant, and does not affect the classical equations of motion of the two-dimensional theory. (On the other hand, the presence of such a term in the path integral merits investigation.) This is reminiscent of another dimensional reduction, that of membranes in D dimensions to strings in $(D-1)$ dimensions, in which the fact that the world volume is changing from 3D to 2D is so special that it allows one to obtain conformal invariance from the three-dimensional membrane diffeomorphisms [13].

Of course the most interesting question at the moment is the quantization of two-dimensional gravity with matter. (Both theories considered here have been quantized successfully in the absence of matter [2,6–11], but the quantization with matter seems a much harder problem.) There are a number of ways of coupling matter to gravity in 2+1 dimensions using the underlying group structure (see, for example, [14]) and one might gain insight into this problem by studying their dimensional reduction to two dimensions. Another interesting problem is to find specific three-dimensional solutions which correspond to the 2D black holes. Finally, one could investigate if there are other rescalings of the three-dimensional generators which yield interesting two-dimensional actions.

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Note added.—After completion of this work, we discovered a paper by Cangemi in which he proposes a different dimensional reduction, starting from a (2+1)-dimensional model based on an Abelian extension of the Poincaré algebra with three extra generators [15]. Subsequently, two more papers have appeared [16] in which more elaborate reduction schemes are proposed. We point out that our formulation is *minimal* in the sense that it is the only one in which the starting point is *pure* 3D gravity.

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