GRAVITATIONAL INSTABILITY OF A MAGNETIZED PLASMA

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(Received October 26, 1961; revised manuscript received November 30, 1961)

The instability of a plasma which is supported under gravity by a magnetic field has earlier been studied by Kruskal and Schwarzschild[1] and by Rosenbluth and Longmire[2] who found that density perturbations were growing at an exponential rate. Later investigations[3,4] indicated that the instability could be suppressed, or at least reduced in its growth rate, for sufficiently small density gradients. A different mechanism for its suppression has recently been based on the effects of the finite Larmor radius.[5]

In the present report the problem will be reconsidered in terms of the fluid equations. We assume a two-dimensional case with a homogeneous, magnetostatic field $\vec{B}$ in the z direction of a rectangular coordinate system, a homogeneous gravitational field $\vec{g}$ in the y direction, and a plasma with the unperturbed density distributions of ions and electrons given by $N(y)$. The perturbed densities of ions and electrons are $n_i = N + \tilde{n}_i$ and $n_e = N + \tilde{n}_e$, where $\tilde{n}_i$ and $\tilde{n}_e$ are small compared to N. The electric field $\vec{E}$ consists of the unperturbed part $\vec{E}_0$ and the perturbation $\vec{\tilde{E}}$. Ions and electrons are assumed to have mass velocities $\vec{v}_i$ and $\vec{v}_e$ situated in the xy plane.

Among the approximations to be made the inertia of electrons will be neglected since the electron mass m_e is much smaller than the ion mass m_i. The center of mass of the plasma is assumed to be at rest in the unperturbed state and only small mass velocities of the ions are considered where $(\vec{v}_i \cdot \vec{\nabla})\vec{v}_i$ is unimportant. Further, we assume small ratios between particle pressure and "magnetic pressure" and small characteristic velocities compared to the speed of light c in such a way that W_i and $W_e \ll V_A \ll c$, where W_i and W_e are the thermal velocities of ions and electrons and V_A is the Alfvén velocity. Then, the influence of the induced magnetic field on the drifts of the density distributions is negligible and the electric field can be expressed by $\vec{\tilde{E}} = -\vec{\nabla}\phi$. We also assume the characteristic length $|N/N'|$ (with $N' \equiv dN/dy$) of the unperturbed density to be much larger than the characteristic lengths $L_x = 2\pi/k_x$ and $L_y = 2\pi/k_y$ of the density perturbation, which in their turn should be much larger than the ion and electron Larmor radii ρ_i and ρ_e. Finally, the unperturbed thermal velocity spectra of ions and electrons are supposed to be peaked around the constant values W_{i0} and W_{e0} and to give rise to scalar pressures $\frac{1}{2}m_i W_{i0}^2 N$ and $\frac{1}{2}m_e W_{e0}^2 N$ which are isotropic in the xy plane. Collisions are neglected. The equivalent magnetic moment of a gyrating particle is then constant at least in first order[6] in the ratios $\rho_i(1/L_x + 1/L_y)$ and $\rho_e(1/L_x + 1/L_y)$. Since B is constant, the "thermal" energies $\frac{1}{2}m_i W_i^2$ and $\frac{1}{2}m_e W_e^2$ as well as ρ_i and ρ_e will then also be constant in the same order in the perturbed state.

With these starting points, conservation of matter and momentum is expressed by

$$\partial n_i/\partial t + \operatorname{div}(n_i \vec{v}_i) = 0; \quad \partial n_e/\partial t + \operatorname{div}(n_e \vec{v}_e) = 0, \qquad (1)$$

and

$$n_i \vec{v}_i = n_i \vec{E}\times\vec{B}/B^2 + m_i n_i \vec{g}\times\vec{B}/eB^2 - \vec{\nabla}p_i \times\vec{B}/eB^2 - m_i n_i(\partial\vec{v}_i/\partial t)\times\vec{B}/eB^2; \qquad (2)$$

$$n_e \vec{v}_e = n_e \vec{E}\times\vec{B}/B^2 - m_e n_e \vec{g}\times\vec{B}/eB^2 + \vec{\nabla}p_e \times\vec{B}/eB^2, \qquad (3)$$

where the scalar pressures are $p_i = \frac{1}{2}m_i W_i^2 n_i$ and $p_e = \frac{1}{2}m_e W_e^2 n_e$. Off-diagonal elements of the pressure tensor which arise from the perturbation have been neglected in Eqs. (2) and (3).

From Maxwell's equations we obtain

$$\operatorname{curl}\vec{B}/\mu_0 = e(n_i\vec{v}_i - n_e\vec{v}_e) + \epsilon_0(\partial\vec{E}/\partial t), \qquad (4)$$

where μ_0 is the magnetic permeability and ϵ_0 the dielectric constant in vacuo. A divergence operation on Eq. (4) yields in combination with Eq. (1)

$$\tilde{n}_i - \tilde{n}_e = -(\epsilon_0/e)\vec{\nabla}^2\phi, \qquad (5)$$

when $\tilde{n}_i - \tilde{n}_e$ and $\vec{\nabla}^2\phi$ are supposed to vanish in the initial state.

The unperturbed electric field is determined from Eqs. (2) and (3) and the condition that the center of mass should be at rest:

$$\vec{E}_0 = (m_i W_{i0}^2/2eN)\vec{\nabla}N - m_i\vec{g}/e, \qquad (6)$$

when the thermal energy of electrons does not exceed that of ions very much. An electromagnetic force $eN(\vec{v}_{i0} - \vec{v}_{e0})\times\vec{B}$ balances the gravitation and pressure forces in the unperturbed state, where $\vec{v}_{i0}$ and $\vec{v}_{e0}$ are corresponding mass velocities.

The drift motion and compression of the density distributions can now be determined[4] from a divergence operation on Eqs. (2) and (3) in combination with Eqs. (1) and (6). If terms up to second order are retained,

$$-\frac{\partial \tilde{n}_i}{\partial t} = \frac{N'}{B}\frac{\partial \phi}{\partial x} + u_f \frac{\partial \tilde{n}_i}{\partial x} - \frac{\epsilon_i}{e}\vec{\nabla}^2 \frac{\partial \phi}{\partial t} - \tfrac{1}{2}\rho_{i0}{}^2 \vec{\nabla}^2 \frac{\partial \tilde{n}_i}{\partial t}, \quad (7)$$

$$-\frac{\partial \tilde{n}_e}{\partial t} = \frac{N'}{B}\frac{\partial \phi}{\partial x} + (u_f - u_{ie})\frac{\partial \tilde{n}_e}{\partial x}, \quad (8)$$

with $u_f = \frac{1}{2}\rho_{i0}{}^2\omega_i N'/N$, $\omega_i = eB/m_i$, $\epsilon_i = Nm_i/B^2$, $u_{ie} = g/\omega_i$, and ρ_{i0} is the unperturbed ion Larmor radius. We observe that $\epsilon_0/\epsilon_i = V_A{}^2/c^2 \ll 1$ and combination of Eqs. (5), (7), and (8) yields

$$\left[\vec{\nabla}^2 \frac{\partial^2}{\partial t^2} - (u_{ie} - u_f + u_\lambda)\vec{\nabla}^2 \frac{\partial^2}{\partial x \partial t} + G\frac{\partial^2}{\partial x^2}\right]\phi = 0, \quad (9)$$

where $u_\lambda = \frac{1}{2}\rho_{i0}{}^2 eN'/\epsilon_i B$, $G = u_{ie}\omega_i N'/N$, and use has been made of the fact that $|N/N'|$ is much larger than the characteristic lengths L_x and L_y.

For perturbations of the form $\exp[i(k_x x + k_y y + \omega t)]$, Eq. (9) gives

$$\omega = \tfrac{1}{2}(\alpha - f + \lambda) \pm \tfrac{1}{2}[(\alpha - f + \lambda)^2 - 4\alpha^2\gamma]^{1/2}, \quad (10)$$

in which $\alpha = k_x u_{ie}$, $f = k_x u_f$, $\lambda = k_x u_\lambda$, and $\gamma = -G/(k_x{}^2 + k_y{}^2)u_{ie}{}^2$.

If α and λ within the parentheses could be dropped, Eq. (10) becomes identical with the result deduced by Rosenbluth et al.[5] for $k_y = 0$. The corresponding stability condition would then become[5]

$$(k_x \rho_i)(\rho_i N'/N) > 4\omega_H/\omega_i, \quad (11)$$

where the growth rate $\omega_H = (\alpha^2\gamma)^{1/2}$ represents that given by the earlier theory.[2] In addition, in the limit of very flat density distributions, i.e., when N' is very small, the frequency would become $\omega \approx -\frac{1}{2}f \pm i\omega_H$.

However, the terms α and λ are of the same order as f and cannot be dropped. From the present definitions $\lambda = f$ and there is an exact cancellation of the term leading to the condition (11). Thus, at least in the present approach, the stability criterion is not given by condition (11) but becomes instead $\gamma < \frac{1}{4}$ as deduced earlier.[3,4] The cancellation is due to the last term in Eq. (7) which represents the inertia of the electric currents produced by the perturbed density gradients. In the limit of small N' the nontrivial solution of the frequency now reduces to $\omega \approx \alpha$.

It remains to investigate whether a treatment of gravitational instability by direct application of the Boltzmann equation, as has been done by Rosenbluth et al.,[5] gives additional information beyond that obtained from the present approach. In particular, Rosenbluth[7] has pointed out that the off-diagonal elements of the pressure tensor give contributions which do not vanish when the relations corresponding to Eqs. (2) and (3) are subject to a divergence operation. This may change the results obtained here.

At the present stage it should only be pointed out that the asymptotic frequency $\omega \approx \alpha = 2\pi u_{ie}/L_x$ for small gradients N' and small driving forces is physically plausible because the density clouds would then have time to move through each other and produce pulsations of the frequency α without changing noticeably during a period L_x/u_{ie}. On the other hand, the earlier solution[5] which leads to the asymptotic frequency $\omega \approx -\frac{1}{2}f \pm i\omega_H$ depends upon N' and cannot be understood in the same way from physical arguments.

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