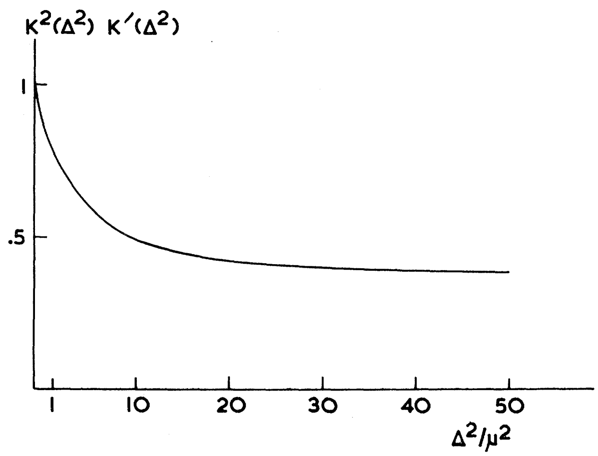

FIG. 3. The function K^2K' and its behavior with the momentum transfer, as obtained by dividing $\psi(\Delta^2)$ [see formula (5)] into $\phi(\Delta^2)$ [see formula (7)] with $\alpha = 60\mu^2$.

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[8]D. S. Chernavskii and I. Dremin, Lebedev Physical Institute, Academy of Sciences of the U.S.S.R., Moscow, Report A-29, 1960 (unpublished).

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[10]Numerical estimates prove the smallness of the term containing $\cos\delta_{33}$ in formula (66) of reference 9. We therefore neglect it here.

[11]The origin of the extra factor KK' appearing in (6) should be clear: K comes from the 3-particle vertex, where the virtual π is emitted, and K' from its propagation.

[12]G. A. Smith, H. Courant, E. C. Fowler, H. Kraybill, J. Sandweiss, and H. Taft, Phys. Rev. 123, 2160 (1961).

[13]All the performed approximations (discussed also in reference 9) could eventually be avoided. The only assumption which is essential in this analysis is the dominance of the 3, 3 resonance. Its validity can be checked experimentally, for instance by comparing with the present theory the Q-value distributions of π^+p from reference 8 at fixed momentum transfer and variable initial energy.

SOLUTIONS OF THE PION-PION PARTIAL WAVE DISPERSION RELATIONS

James S. Ball and David Y. Wong
University of California at San Diego, La Jolla, California
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In 1958, Chew and Mandelstam obtained a set of integral equations for the pion-pion partial wave amplitudes.[1] The underlying physical assumption of these equations is that the exchange of a pion pair in pion-pion scattering should constitute the longest range force which, in turn, dominates the scattering amplitude at low energy. Since that time, a number of authors have published various approximate solutions to these equations based on the further assumption that the major part of the two-pion force comes from the relative S and P states of the exchanged pair.[1-5] The remaining portion of the two-pion force as well as the multipion forces are usually represented by phenomenological parameters. Unfortunately, there are mathematical difficulties even in handling the P-wave exchange force which necessitates some cutoff procedure. In principle, such cutoff procedures also require adjustable parameters. It is not clear from previous results what is the minimum number of parameters required for a realistic solution of the low-energy pion-pion problem. The purpose of this Letter is to clarify this point and to present a few numerical solutions which fall within the uncertainty of the present experimental data.[6]

The coupled S- and P-wave pion-pion dispersion equations[1] are solved numerically including left-hand cuts that satisfy crossing symmetry up to a cutoff point. The remaining left-hand cut of each partial wave amplitude is replaced by a pole together with a subtraction constant at the symmetry point $\nu = -\frac{2}{3}$, where ν is the center-of-mass momentum squared in pion units. The solutions obtained are found to depend sensitively on three parameters, and no further reduction in the number of parameters seems possible if only low partial waves are kept in all absorptive amplitudes.

The S-wave subtraction constants are related by the 5 to 2 ratio:

$$\lambda \equiv -\tfrac{1}{5}A_{l=0}^{I=0}(-\tfrac{2}{3}) = -\tfrac{1}{2}A_{l=0}^{I=2}(-\tfrac{2}{3}).$$

The residue of the P-wave pole is determined by the requirement that the P-wave amplitude vanish at the threshold while the position of this pole is chosen to fit a resonance energy within the range 700-800 Mev ($5.2 < \nu_r < 7.2$).[6] The residues of the S-wave poles ($I = 0, 2$) are determined by the crossing condition which relates the derivatives of the S-wave amplitudes to the value of the P-wave amplitude at the symmetry point:

$$a_1 \equiv \tfrac{3}{2} A_{l=1}^{I=1}(-\tfrac{2}{3})$$

$$= \tfrac{1}{6} A'{}_{l=0}^{I=0}(-\tfrac{2}{3}) = -\tfrac{1}{3} A'{}_{l=0}^{I=2}(-\tfrac{2}{3}).$$

The positions of the S-wave poles are left as free parameters of the problem. However, for a given set of (λ, a_1, ν_r), the values and derivatives of all three partial wave amplitudes at the symmetry point are essentially fixed, and the higher derivatives are mainly controlled by the "near-by" left-hand cut. Hence the solution is insensitive to either the position of the S-wave poles or the cutoff point of the "near-by" left-hand discontinuity. In fact, a factor of two variation in these parameters affects the solution in the physical region by less than 5-10%. Thus our result is basically a three-parameter solution, (λ, a_1, ν_r).

In order to fit the width of the P-wave resonance as indicated by recent experiments,[2] we choose $0.02 < a_1 \lesssim 0.05$. For a given pair of (a_1, ν_r), λ is varied from one extreme where an $I = 0$ bound state occurs to the other extreme where the $I = 2$ ghost appears in the "near-by" left-hand region. We find that λ lies within the range ± 0.5. At present, there is no decisive experimental result on S-wave pion-pion interaction. Therefore, we only present a few typical solutions as shown in Figs. 1 and 2.

It turns out that the P-wave phenomenological pole is located quite far out on the left ($-\nu \gtrsim 10^3$) but still plays an important role in making the resonance. For the S waves, the effect of the poles in the physical region is fairly small; however, the solution does depend strongly on the subtraction constant. We therefore conclude that the "near-by left-hand cut" does not dominate even the low-energy part of the scattering in either the P or the S state. Hence the number of independent parameters cannot be reduced within the framework of keeping only low partial waves in all absorptive amplitudes.

We should mention that our solutions are qualitatively quite similar to those given by Desai,[3] who approximated the entire left-hand cut of each partial wave amplitude by a pole. Of course, our combined treatment of polology and "near-by" left-hand cuts should yield more detailed information on the structure of the amplitudes. Incidentally, we find that our solutions in general do not satisfy Desai's additional symmetry point condition involving second derivatives of the S-wave amplitudes and the first derivative of the P-wave amplitude. But we do not believe that this condition can be used to select a smaller set of solutions since the condition itself is inexact. In fact, our solutions deviate from this condition the most when the $I = 0$ contribution to the near-by left-hand cut is large. This is precisely the circumstance

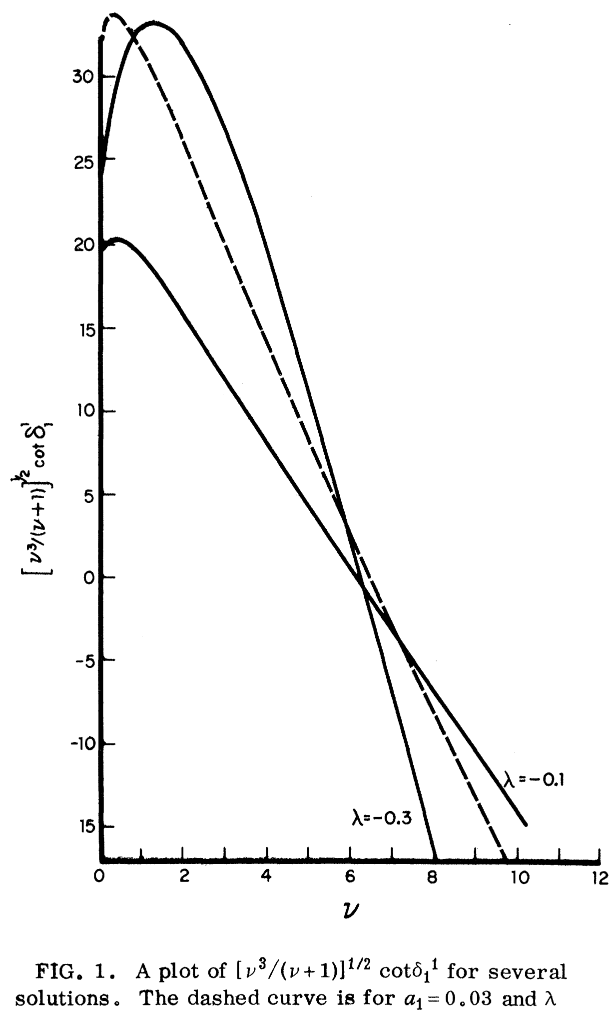

FIG. 1. A plot of $[\nu^3/(\nu+1)]^{1/2} \cot\delta_1{}^1$ for several solutions. The dashed curve is for $a_1 = 0.03$ and $\lambda = -0.10$. The solid curves are for $a_1 = 0.05$ with $\lambda = -0.10$ and $\lambda = -0.30$. The curvature in the neighborhood of $\nu = 0$ reflects the effect of the S wave in the crossed channel.

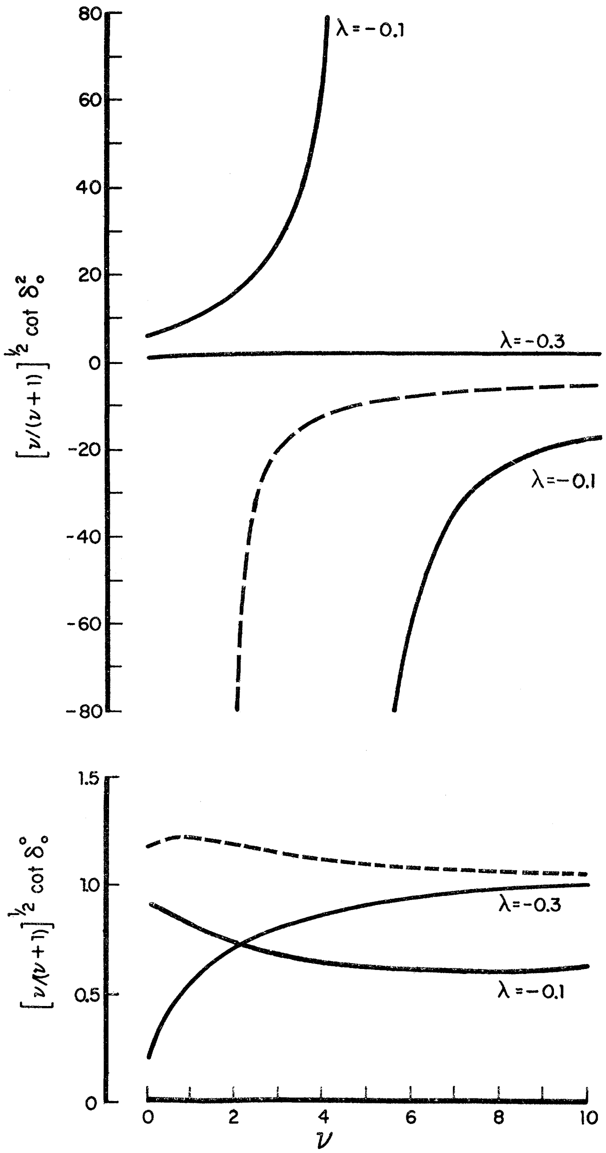

FIG. 2. A plot of $[\nu/(\nu+1)]^{1/2}\cot\delta_0^0$ and $[\nu/(\nu+1)]^{1/2} \times \cot\delta_0^2$ for various values of the parameters. The dashed curves are for $a_1=0.03$ and $\lambda=-0.10$. The solid curves are for $a_1=0.05$ with $\lambda=-0.10$ and $\lambda=-0.30$.

under which derivatives of the higher partial waves neglected in Desai's condition are not small. Since the higher derivatives are almost completely controlled by the "near-by" portion of the branch cuts, any valid symmetry point condition involving higher derivatives will automatically be satisfied by solutions which contain "near-by" cuts given by crossing symmetry.

At this point we would like to clarify some confusion on a recent paper by Bransden and Moffat.[4] In their work, it appears that the S-wave subtraction constant alone (λ) is sufficient for the determination of both the S- and P-wave amplitudes. First of all, we should point out that the left-hand discontinuity of the scattering amplitude calculated from Bransden and Moffat's solution does not agree with the left-hand cut they obtained by using crossing symmetry. This discrepancy is due to a mathematical error in their iterative procedure which leads to the wrong branch of the quadratic relation between real and imaginary parts of a function and its inverse beyond $\nu \simeq -15\,\mu^2$. In fact, the left-hand cut in their solution is practically zero for $\nu \lesssim -15\,\mu^2$, whereas the left-hand cut calculated by crossing approaches a fairly large constant (~10).[7] This explains the apparent insensitivity of their solution to the cutoff and the formal convergence of their iteration even without a cutoff. However, it is clear that the cutoff is in effect present in their solution. Now, a more important question to be asked is whether the inverse amplitude of Bransden and Moffat contains any zero on the left. The answer is that if the left-hand discontinuity is literally cut off anywhere within the range $-500 \lesssim \nu < -1$, then there is a zero in the inverse function of the P-wave amplitude in the neighborhood of $\nu \sim -500$. If the scheme of Bransden and Moffat is carried out without a cutoff, then this zero will be displaced slightly from the negative real axis. In fact, this "far-away" pole in the P-wave amplitude is the major contributor to the resonance as we would expect. The position and residue of this pole are nearly independent of the left-hand cut and almost totally determined by two parameters a_1 and ξ_1 which were eventually adjusted to fit the S-wave derivative conditions at the symmetry point. It is clear that one can vary the position and width of the resonance substantially by varying a_1 and ξ_1. The derivative conditions can be satisfied by retaining a reasonable medium-range left-hand cut in each of the S-wave amplitudes. These cuts in the S-wave amplitudes are certainly no more unnatural than the far-away-yet-dominating pole in the P-wave amplitude. By allowing a variation in (a_1, ξ_1), one would end up with a three-parameter solution which would be essentially the same as ours.

In a recent paper by Zachariasen,[5] it also appears that the position and width of the P-wave resonance can be roughly determined without introducing any arbitrary parameter (aside from parameters for the S-wave amplitudes which he did not consider). Again, one can show that the far-left portion of the cut is the major contributor to the resonance in Zachariasen's solution. Here, the left-hand cut is approximated by the Born term of a vector-meson exchange diagram divided by a D function which makes his result unitary. This D function becomes quite large in the far-left region and hence strongly suppresses the single vector-meson cut. This particular choice of the far-away left-hand cut certainly has not been justified on any theoretical basis. Thus there is ample room for doing phenomenological adjustments on the far-away left-hand cut which, we stress, could change both the position and the width of the resonance by a large amount.

So far, we have seen that "near-by" left-hand cuts do not play a very significant role in all semi-phenomenological solutions of the low-energy S-P wave pion-pion problem. This apparently contradicts the speculation that the lowest mass state provides the longest range force which dominates the low-energy scattering. However, the failure might well lie in the intuitive correspondence between near-by left-hand cuts and long-range forces.[8] It is true that partial wave expansions on the left require a cutoff, but the cutoff procedure may also cause the truncation of an important part of the long-range force which must be reinstated by phenomenological cuts or poles.

At present, it seems to us that an improved treatment of the left-hand cut may be achieved by using Regge's integral representation of the scattering amplitude instead of the conventional partial wave expansion.[9] In Regge's representation, the P-wave resonance is associated with a pole in the complex angular momentum plane. This pole produces an oscillating far-away left-hand cut as well as the usual near-by cut. As long as this pole does exist, there is no question that the oscillating cut should be included in the solution. This may well explain why far-away cuts in all phenomenological solutions seem to be quite important. Thus one may hope to reduce the number of phenomenological parameters by including the oscillating portion of the cut.

Finally, we would like to remind the reader that so far we have not taken into account inelastic scattering in the physical region. However, due to the unitarity restriction, production processes are expected to produce fairly small effects on the elastic scattering amplitude except under situations similar to those discussed by Frazer and one of us (J.S.B.).[10] The prospect of this latter type of resonances in the pion-pion system seems to be rather small.

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