REMARK ON THE ALGEBRA OF INTERACTIONS

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It is the purpose of this note to point out some remarkable similarities between the structure of interactions and the algebra of octonions.

To characterize briefly this strange algebra, we recall some familiar properties of quaternions.

$$X = X_0 e_0 + X_\alpha e_\alpha \tag{1}$$

(summation over $\alpha = 1, 2, 3$) is a quaternion if

$$e_0^2 = 1, \qquad e_0 e_\alpha = e_\alpha e_0,$$

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}, \qquad e_\alpha e_\beta = \epsilon_{\alpha\beta\gamma} e_\gamma. \tag{2}$$

$\epsilon_{\alpha\beta\gamma}$ is totally antisymmetric; $\epsilon_{123} = 1$. If the "components" X_0, X_α are real, define

$$(X, Y) = \tfrac{1}{2}(\bar{X}Y + \bar{Y}X), \quad \bar{X} = X_0 e_0 - e_\alpha X_\alpha, \tag{3}$$

$$N(X) = (X, X). \tag{4}$$

In this real case the norm of the product XS of two quaternions satisfies

$$N(SX) = N(XS) = N(X)N(S), \tag{5}$$

from which by associativity

$$N(S^{-1}XS) = N(X). \tag{6}$$

A null quaternion is defined by

$$X = 0: \; X_0 = X_\alpha = 0. \tag{7}$$

From Eqs. (2) and (3), $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ so that the three e_α are like orthogonal unit vectors.

Quaternions share with real and complex numbers the property (5). There exists only one other number system[1] (with two variants) for which Eq. (5) is true, namely, the octonions. The first variant satisfies in fact all Eqs. (1)-(7)—with two changes of meaning, however: (a) $\alpha, \beta, \cdots = 1, \cdots, 7$; (b) $\epsilon_{\alpha\beta\gamma}$ is totally antisymmetric and equals +1 for

$$(\alpha\beta\gamma) = (123),\ (145),\ (167),\ (246),\ (275),\ (365),\ (374). \tag{8}$$

Thus one generates octonions from our starting quaternion by introducing a further unit vector e_4, $(e_\alpha, e_4) = 0$, $\alpha = 1, 2, 3$. Then by vector product formation $2e_5 = e_1 \times e_4$, $2e_6 = e_2 \times e_4$, $-2e_7 = e_3 \times e_4$. Equations (2) and (8) show that octonion multiplication is nonassociative and hence does not yield a group.[2] However, any two octonions generate a group; hence Eq. (6) is true as well.

The continuous automorphisms of the e_α form the group G_2. In addition we note the following two discrete operations.

$$P_1: \; e_0 \to -e_0, \tag{9}$$

$$P_2: \; e_\alpha \to ie_\alpha; \quad \alpha = 4, 5, 6, 7; \; i = (-1)^{1/2}. \tag{10}$$

P_1 does not, P_2 does change the multiplication table (8). P_2 adjoins[3] to an X a "split" $X^{(s)}$, $X^{(s)} = X_{\rm re} + iX_{\rm im}$, where $X_{\rm re}$, $X_{\rm im}$ are orthogonal. Define $N'(X^{(s)}) = N(X_{\rm re}) - N(X_{\rm im})$; then

$$N'(X^{(s)}Y^{(s)}) = N'(X^{(s)})N'(Y^{(s)}),$$

an alternative version of Eq. (5) (second variant).

q-octonions have q-number fields as components. Here too one can define an inner product by taking the Hermitian average on the right side of Eq. (3). Let X be a q-octonion, S a real c-octonion. One shows that Eqs. (5) and (6) are again true. Thus if $N(S) = 1$, then $N(XS) = N(SX) = N(S^{-1}XS) = N(X)$, i.e., one can define octonion gauge transformations of q-octonions.

Let $B = B_0 - ie_\alpha B_\alpha$, $M = M_0 - ie_\alpha M_\alpha$, where the components of $B(M)$ are spin $\frac{1}{2}(0)$ fields. The equation[4]

$$(\gamma\partial + m)B + iG\gamma_5 BM = 0 \tag{11}$$

expresses, by Eqs. (2), (7), and (8), the strong interactions with a high symmetry provided $B_0 = \Lambda$, $2^{1/2}B_1 = \Sigma^+ + \Sigma^-$, $i2^{1/2}B_2 = \Sigma^- - \Sigma^+$, $B^3 = \Sigma^0$, $i2^{1/2}B_4 = n + \Xi^0$, $-2^{1/2}B_5 = p + \Xi^-$, $i2^{1/2}B_6 = p - \Xi^-$, $-2^{1/2}B_7 = n - \Xi^0$. Put $B = B(\Lambda, \Sigma, N, \Xi)$. Then[5]

$$M = B(\sigma, \pi, K, K^G). \tag{12}$$

The M components are eigenstates of charge conjugation. The "full symmetry" is: global symmetry of the type[6] G^- and moreover K-coupling strengths = minus π-nucleon coupling.

Thus the octonion calculus which involves sets of eight "equivalent particles" automatically produces all needed selection rules plus those unwanted extra ones implied by too strong a symmetry. We now observe that we can adjoin to B two other octonions, namely $B^{(1)}=P_1B$ and $B^{(2)}=P_2RB$, where $R=(e_7\to e_4,\ e_4\to -e_7,\ e_5\to e_6,\ e_6\to -e_5)$ and is in G_2. Replace in Eq. (11) BM by $[B+c_1B^{(1)}+c_2B^{(2)}]M$ and the following happens: In the right qualitative way, the baryon masses split as $8=1+3+2+2$, the meson masses as $8=1+3+4$. Each of the three couplings separately have equivalent full symmetry, denoted by F, $F^{(1)}$, $F^{(2)}$, respectively. Some partial symmetries are: $F+F^{(2)}=$ doublet approximation, $F+F^{(1)}=$ Behrends-Sirlin scheme.[7] Note that three is the minimal number of clashing full symmetries which breaks the degeneracies.[8] The usual charge operator

$$Q=T_3+Y_3 \tag{13}$$

is the sum of the third components of isotopic spin and hyperspin and has the following curious property. With respect to each of the F's separately, Q itself is isomorphic to a third component of angular momentum. (This is not true for any other nontrivial linear combination of T_3 and Y_3.)

The above example of the lifting of degeneracies is not unique. In particular, the present calculus is not tied to the strange particle parities.[9]

Consider next the transformations

$$B\to S^{-1}BS,\quad M\to S^{-1}MS, \tag{14}$$

with S a real c-octonion, $N(S)=1$. By application of G_2, S can be brought into the canonical form $S=\exp e_4\xi$. For infinitesimal ξ, the transformation (14) is not in G_2. Even so, Eq. (11) with $G=0$ is invariant under Eq. (13) as S, B generate a group. For $G\neq 0$, Eq. (11) becomes $(\gamma\partial+m)B+S[(S^{-1}BS)\times(S^{-1}MS)]S^{-1}=0$. For "infinitesimal" ξ one finds that now $BM\to BM+$"weak" interaction with $\Delta T=\frac{1}{2}$. Thus e_4 acts as the spurion. For the present we do not discuss the parity structure of weak nonleptonic interactions so generated.

If the octonion algebra envelops in some sense the structure of interactions, one may ask how the leptons could enter. Here the electric charge should form one bridge. Especially if two neutrinos ν_1, ν_2 exist, it is interesting to contemplate the possibility that, rather like M, the lepton is a self-charge conjugate octonion L. L can be so constructed that, where $(B, Q\gamma_\lambda B)$ is the electric baryon current, so $(L, Q\gamma_\lambda L)$ is the electric lepton current. In treating neutral leptons considerable arbitrariness exists, however, partly connected with the group $SU(3)$ (subgroup of G_2) of automorphisms which keep one e_α fixed. Lepton conservation induces insufficient restrictions. Some partial results follow.

Using doublet language,[10] write B as $B(N_1, N_2, N_3, N_4)$. Put[11]

$$L(\omega)=2^{-1/2}B(\omega n_4^{\,c},\ -\omega n_3^{\,c}, n_3, n_4), \tag{15}$$

$$n_3=\begin{pmatrix}\nu_1\\ e\end{pmatrix},\quad n_4=\begin{pmatrix}\nu_2\\ \mu\end{pmatrix}.$$

ω is a 2×2 unitarity matrix; $(\omega^\dagger\tau_3\omega)^T=-\tau_3$. As for B, one checks that Q is indeed the L electric charge operator.

Define current octonions[12] $J=\bar{L}\gamma L$, $j=\bar{B}\gamma(1+\alpha\gamma_5)\times B$. The inner product,

$$(J,j)=\sum_{a=0}^{7}J_aj_a,$$

describes leptonic transitions, as follows. $a=0,3$: $\Delta S=0$, neutral; $a=1,2$: $\Delta S=0$, charged; $a=4,7$: $|\Delta S|=1$, neutral; $a=5,6$: $|\Delta S|=1$, charged. One finds $J_0=J_4=J_7\equiv 0$ for any ω as defined. Hence there are no neutral $|\Delta S|=1$ transitions. Let $J^{(1)}$ correspond to $\omega=i\tau_1$. One finds $J_5^{(1)}=J_6^{(1)}=0$. Thus $(J^{(1)}+j, J^{(1)}+j)$ describes the universal $\Delta S=0$ interaction.[13] Let $J^{(2)}$ refer to (a) interchange of ν_1 with ν_i^c, ν_2 with ν_j^c, $(i\neq j)=1,2$; (b) $\omega=i\tau_1$. One finds $J_1^{(2)}=J_2^{(2)}=0$, while now $J_5^{(2)}, J_6^{(2)}$ are lepton-conserving $|\Delta S|=1$ charged currents. The two choices for (i,j) are related to the neutrino-flip question.[14] It is perhaps a good aspect that the absence of $|\Delta S|=1$ neutral leptonic follows from a specific algebra. It is perhaps a bad aspect that the synthesis of $\Delta S=0$ leptonic and $|\Delta S|=1$ is so far not unique. (The use of different L's is not unlike the use of different B's for strong interaction asymmetries.)

It should be noted that all results stated here could have been written without using octonions at all. The formal structure here described seemed sufficiently intriguing to communicate, however. Yet the close connection between octonion algebra and some aspects of the interactions may be nothing but a strange coincidence. To be more than that, this algebra should play a dynamical role.[15]

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[1]Hurwitz' theorem. For early developments see L. Dickson, Ann. Math. 20, 155 (1918). For both variants the components are real (real octonions).

[2]But rather a so-called quasi-group where the associative law is replaced by a certain weaker law. For the present it suffices to state that for octonions this law is: The associator $X(YZ) - (XY)Z$ is totally antisymmetric in X, Y, Z.

[3]In this case and also for the q-octonions mentioned below, we have no longer a division algebra. This does not matter as we need not take inverses of wave fields.

[4]For the purpose of exposition we consider only trilinear interactions and leave aside the question whether strong interactions are generated by gauge fields.

[5]σ is a $T=S=0$ meson conjectured by many authors. $\sigma \equiv 0$ is not necessarily excluded. $K=(K^+, K^0)$; $K^G = (-\bar{K}^0, K^-)$.

[6]In a terminology employed elsewhere; A. Pais, Phys. Rev. 122, 317 (1961).

[7]R. Behrends and A. Sirlin, Phys. Rev. 121, 324 (1961).

[8]Example: $B^{(3)} = R'B$; $R' = (e_\alpha \to -e_\alpha)$, $\alpha = 4,5,6,7$ and is in G_2. Even with all these couplings there remains a simple π-coupling constant relation: $(NN\pi) = -(\Sigma\Sigma\pi) = (\Xi\Xi\pi)$. Note: P_1 corresponds essentially to $BM \to MB$.

[9]Example: Write the above B as $B(N, \Lambda, \Sigma, \Xi)$, define $B' = B(N, c_1\gamma_5\Lambda, \Sigma, c_2\gamma_5\Xi)$, and replace in Eq. (11) BM by $B'M$. This spreads the masses due to space parity effects. The curious property of Q is now lost. The example in the text may be said to generate splits due to two "isotopic parities."

[10]In the notation of reference 6. As is done in part of the quoted paper, the doublets are used as a mathematical device without necessarily insisting on any mass degeneracy.

[11]C = charge conjugate. $\nu_i = \frac{1}{2}(1+\gamma_5)\psi_i$. ψ_i is a massless spinor. It is possible but unattractive to put $\nu_1 \equiv \nu_2$. T = transpose.

[12]The four-vector index is suppressed. J will always have the correct V-A combination due to the very definition of ν_i. The j_α, $\alpha = 4,5,6,7$, have $\Delta T = \frac{1}{2}$. Also $\Delta T = \frac{3}{2}$ currents can be constructed, namely by using G_2 generators; see reference 7.

[13]Including a neutral $\Delta S = 0$ current $J_3^{(1)}$ without μe terms. To such a current there is so far no objection; see S. Bludman, Phys. Rev. 115, 468 (1959). (In the present case the $\bar{\mu}\mu$ and $\bar{e}e$ terms moreover conserve parity.) For a first attempt to tie lepton phenomena to a group, see S. Bludman, Nuovo cimento 9, 433 (1958).

[14]G. Feinberg, F. Gürsey, and A. Pais, Phys. Rev. Letters 7, 208 (1961). Another ambiguity exists due to the fact that j need not necessarily be the same for $\Delta S = 0$ as for $\Delta S = 1$.

[15]Note the possibility of linearized octonion wave equations. Let X be an octonion field, $O = e_\mu p_\mu + ie_\rho m_\rho + m_0$, $\mu = 1$-4, $\rho = 5$-7, m_ρ mass parameters. Put $OX = 0$. Then $\bar{O}(OX) = (\bar{O}O)X = 0$. Hence we get a standard wave equation for mass $(m_\rho{}^2 + m_0{}^2)^{1/2}$, for each component of X.