greater than either of the corresponding values of $N(p)$.

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POSSIBLE EXPLANATION OF THE HIGHER PION-NUCLEON AND K^--p RESONANCES IN TERMS OF INELASTIC THRESHOLDS

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We have found by means of partial-wave dispersion relations that a large, rapidly rising inelastic cross section, such as is observed in the vicinity of the second and third pion-nucleon resonances,[1] can account for a sharp peak in the elastic scattering. We have calculated the inelastic contribution to the higher partial waves by means of the strip approximation to the Mandelstam representation,[2,3] in which the principal mechanism is the production of the $J=1$, $I=1$ pion-pion resonance. Assigning to this resonance a position and width in accord with recent experimental evidence,[4] we find strong inelastic scattering in angular momentum and isotopic spin states which phenomenological analyses have suggested as accounting for the resonances.[5,6] Moreover, resonances seem to be implied in approximately the correct positions, although we are unable at this stage to make really quantitative calculations. The same strip approximation calculation seems to imply a K^--p resonance, primarily in the $D_{3/2}$ state, at the observed energy,[7] provided the recently observed K^* has $J=1$ and $I=\frac{1}{2}$.[8]

In order to find the effect of strong, rapidly rising inelastic scattering in a given partial wave, consider the amplitude $f(\nu)$, which assumes the following form in the physical region:

$$f(\nu)=(e^{2i\delta}-1)/2i. \tag{1}$$

Above the inelastic threshold ν_i, $\delta=\delta_R+i\delta_I$, with $\delta_I>0$ according to unitarity. The variable ν is chosen appropriate to each particular reaction; for equal-mass scattering $\nu=k^2$, the square of the momentum in the center-of-mass system. Note that

$$\operatorname{Im} f=|f|^2+(1-\eta^2)/4, \tag{2}$$

where $\eta=e^{-2\delta_I}$. Thus the inelastic contribution to the cross section is proportional to $(1-\eta^2)/4$, so that δ_I is determined if the inelastic scattering in the partial wave is known. Let us assume for the moment that this is the case.

In order to keep the argument from being obscured by inessential complications, we shall now make two simplifying assumptions which could be relaxed without difficulty. We assume that $A(\nu)=f(\nu)/k$ is the function which is analytic in the ν plane except for the singularities given by the Mandelstam representation, thereby neglecting some relativistic effects. Since the point we wish to make concerns the effect of the inelastic scattering on the elastic at rather high energies, we also neglect the unphysical cuts.

We now have the following mathematical problem: Find a function $A=f/k$ with the properties: (a) It is analytic in the entire ν plane, except for a cut along the positive real axis. (b) It is real on the negative real axis. (c) The function $f(\nu)$ assumes the form of Eq. (1), with given δ_I, when the positive real axis is approached from above.

The solution is remarkably simple; construct

$$\delta(\nu)=\frac{k}{\pi}\int_{\nu_i}^{\infty}d\nu'\frac{\delta_I(\nu')}{k'(\nu'-\nu)}. \tag{3}$$

Using this formula to define δ in the entire ν plane, we assert that Eq. (1) provides a solution to the problem posed.[9,10] It can easily be verified that $A(\nu)$ has only the desired branch cuts, provided k is defined to have its cut along the positive real axis.

Let us now consider the effect upon the elastic amplitude of a rapidly rising inelastic cross section which rises to a value near its unitarity limit, such as is shown by the solid line in Fig. 1. Equation (3) implies that δ_R will be a sharply peaked function in the neighborhood of the rise

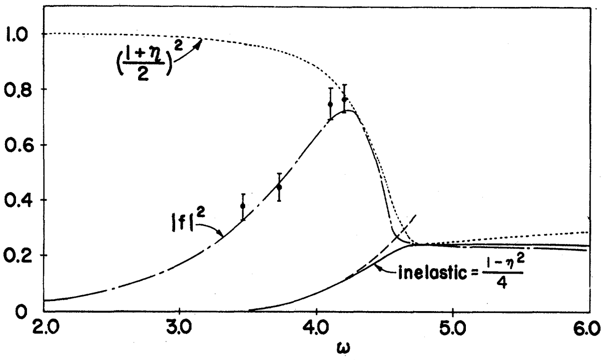

FIG. 1. A fit to the $D_{3/2}$ pion-nucleon resonance. The dashed line is the calculated $\sigma_I(D_{3/2})$. The dash-dot line is the absolute square of the elastic amplitude calculated from the inelastic cross section shown by the solid line. This inelastic cross section, which joins smoothly with our calculated values, has been chosen arbitrarily to produce a fit to the observed elastic scattering.[1] The dotted line is the unitarity limit for the absolute square of the elastic amplitude. The variable ω is defined as $\omega = W - m$, where W is the c.m. total energy and m is the nucleon mass.

in the inelastic cross section. The resulting elastic amplitude for a particular choice of δ_I is shown in Fig. 1. Note that the trailing edge of the peak is controlled mainly by the unitarity limit. The height and exact shape of the peak depend not only upon the rate of rise of the inelastic cross section, but also on its behavior at higher energies. It is possible, as illustrated in Fig. 1, to find reasonable inelastic cross sections which imply elastic peaks sharp enough to fit the higher resonances in pion-nucleon scattering.[11] The inelastic cross section itself need not be sharply peaked to produce a sharp, sizeable peak in the elastic amplitude. This statement is in principle subject to experimental verification; for example, by measuring the absorption in the $D_{3/2}$, $I=\frac{1}{2}$, pion-nucleon state above the second resonance. Note that our results do not imply a large elastic peak at all inelastic thresholds; the condition of a rapid rise to near-total absorption must be satisfied.

We turn now to the second part of the problem; namely, to calculate the inelastic cross section in each pion-nucleon partial wave and thereby obtain a value for δ_I to substitute in Eq. (3). As a first approximation to the problem, we use the "strip approximation" to the Mandelstam representation, in which only those parts of the double spectral functions corresponding to the lowest values of the momentum transfer are calculated.[2,3]

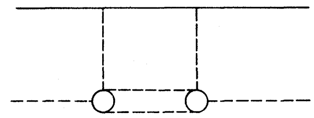

FIG. 2. Diagram giving rise to the pion-nucleon spectral function in the strip approximation.

The formulas, which are obtained by applying unitarity to the Mandelstam representation,[12] correspond to the process shown in Fig. 2. For the π-π amplitude we use a $J=1$, $I=1$ resonance formula.[13,14] Having calculated the double spectral functions for low momentum transfer, we substitute them into the equation for the imaginary part of the scattering amplitude,

$$\mathrm{Im}A(s,t) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{a(s,t)}{t'-t}, \tag{4}$$

where the lower limit t_0 is given by the Mandelstam boundary curve. We calculate the spectral function $a(s,t)$ correctly out to the boundary curve corresponding to four-pion exchange. The effect of neglecting these higher-mass-exchange contributions will be small for a sufficiently high partial wave. The $J=\frac{1}{2}$ states will certainly be unreliable, since the integral in Eq. (4) diverges logarithmically if we approximate the pion-pion amplitude by the P wave alone. For these states the contributions we have neglected, such as the contribution of the (3, 3) resonance, will surely be important. Therefore our calculation has some chance of reliability only for $J \geq \frac{3}{2}$. We project out partial waves from Eq. (4) and equate our calculated value of $\mathrm{Im}f_l$ with $(1-\eta^2)/4$, since the scattering via intermediate two-pion state is inelastic. Let us introduce the notation $\sigma_I(D_{3/2})$, for example, for the quantity $(1-\eta^2)/4$ in the $D_{3/2}$ state. Note that this quantity differs from the actual inelastic cross section by a factor $4\pi(J+\frac{1}{2})/k^2$.

Our results for these states are as follows: $\sigma_I(D_{3/2})$ rises rapidly in the region of the "threshold" for production of a π-π resonance, but rises to a height exceeding the unitarity limit by almost an order of magnitude. Our approximation has broken down because we have not taken unitarity into account for the processes $\pi+N \to \pi+\pi+N$ and $\pi+\pi+N \to \pi+\pi+N$. It may be possible to remedy this defect by means of an approximate treatment of these processes in which the particles are allowed to interact only in pairs. Such a calculation is now in progress. If we make the plausible conjecture that in a correctly unitarized calculation our large $\sigma_I(D_{3/2})$ will remain sizeable and will continue to rise rapidly, we can make a rough calculation of the position of a peak in the elastic $D_{3/2}$ amplitude. With a π-π resonance energy around $5m_\pi$, this peak coincides with the second resonance. Moreover, the effect of unitarity should not affect the results of our present calculation greatly as long as the result is well below the unitarity limit. Hence we can calculate the inelastic cross section in the region where it is beginning to rise and therefore, according to the arguments above, we can roughly predict the steepness of the trailing edge of the elastic peak. In Fig. 3 these results are compared with experiment. Also shown is the inelastic contribution to the only other state with $J \geq \frac{3}{2}$ which is sizeable in our model, the $F_{5/2}$ state. It rises rapidly at an energy coinciding with the third resonance. These angular momentum assignments seem to be consistent with experiment.[5] The way they arise in our model can be understood quite easily: Just above the "threshold" for production of a π-π resonance, the state in which the nucleon and the π-π resonance are in a relative S wave dominates. This state couples with the pion-nucleon system in the $D_{3/2}$ and $S_{1/2}$ states. We are unable to calculate $\sigma_I(S_{1/2})$ reliably, but we do expect $\sigma_I(D_{3/2})$ to rise rapidly in this region.[15] At a higher energy (thus explaining the separation between the two resonances), the relative P wave of the nucleon and π-π system will become important. This couples to the pion-nucleon $P_{1/2}$, $P_{3/2}$, and $F_{5/2}$ states. Again we cannot calculate $\sigma_I(P_{1/2})$. We find that $\sigma_I(F_{5/2})$ does indeed become important in our calculation, but $\sigma_I(P_{3/2})$ does not seem to attain a sufficiently high value to produce a large effect on the elastic scattering. This numerical result could be changed by a more definitive calculation.

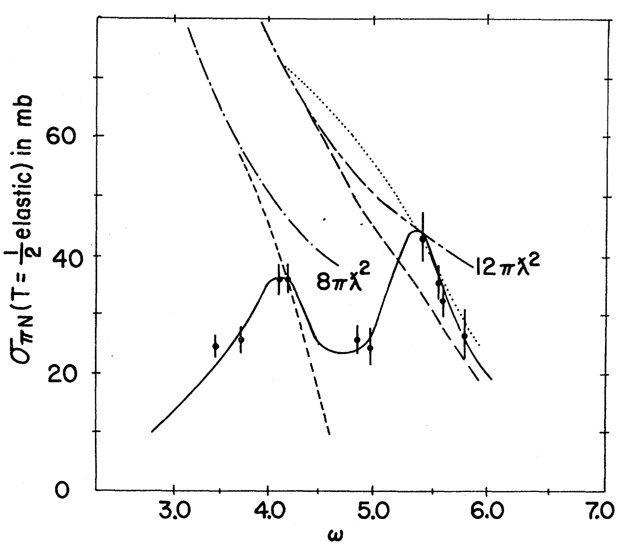

FIG. 3. The second and third pion-nucleon resonances. The dashed lines are the unitarity limits for elastic scattering in the $D_{3/2}$ and $F_{5/2}$ states calculated from the double spectral functions. The dotted line is the unitarity limit if a background of a saturated $\sigma_I(D_{3/2})$ is added. The points are from Falk-Vairant and Valladas.[1]

The isotopic spin dependence of the diagram in Fig. 2 is such that the $I=\frac{1}{2}$ state is favored over the $I=\frac{3}{2}$ by a factor of four to one. This factor reduces $\sigma_I(F_{5/2})$ sufficiently that no analog of the third resonance is expected in π^+-p scattering. However, $\sigma_I(D_{3/2})$ is still large enough that the elastic amplitude in this state should show a peak around 800 Mev in π^+-p scattering. This may be the knee in the cross section which Carruthers has identified with the $D_{3/2}$ state.[6]

We have made a similar calculation for K^--p scattering around 1 Bev/c lab momentum, where a peak in the cross section has recently been found.[7] Again we calculate the diagram in Fig. 2, but now the incident and final mesons are K^-, and the intermediate resonance is the K^* resonance found by Alston et al.[8,16] If we assume that the K^* has $J=1$ (experiments have not yet distinguished between $J=0$ and $J=1$), then we again find a strong, rapidly rising $\sigma_I(D_{3/2})$ at just the right position to correspond to the observed peak (see Fig. 4). This implies a nearly saturated $D_{3/2}$ contribution to K^* production at 1.15 Bev/c, the energy in the experiment of Alston et al. Such a contribution to the process $K^-+p \rightarrow \bar{K}^0+\pi^-+p$ turns out to be only 1.0 mb, however, which is not in disagreement with the experimental value which is of the order of 1 mb. In this process $\sigma_I(F_{5/2})$ does not come out to be strong, largely because of the narrowness of the K^* resonance as compared to the π-π resonance. We see from Fig. 4 that with a reasonable background the $D_{3/2}$ state does not seem to be able to account for the entire peak. The difference is probably due to the $S_{1/2}$ state which we are unable to calculate reliably.

If we assume the K^* has $J=0$, we find no large amplitudes with $J \geq \frac{3}{2}$, because a nucleon and K^* in a relative S state can then couple only to $J=\frac{1}{2}$ (which is clearly ruled out by the height of the observed peak). The relative P state does not contribute strongly for this process. Thus we definitely require the K^* to have $J=1$ if our model is to be valid. If we further assume $I=\frac{1}{2}$, as indicated by the K^* experiment, we find that in K-N scattering the $I=0$ state is favored over $I=1$ by a

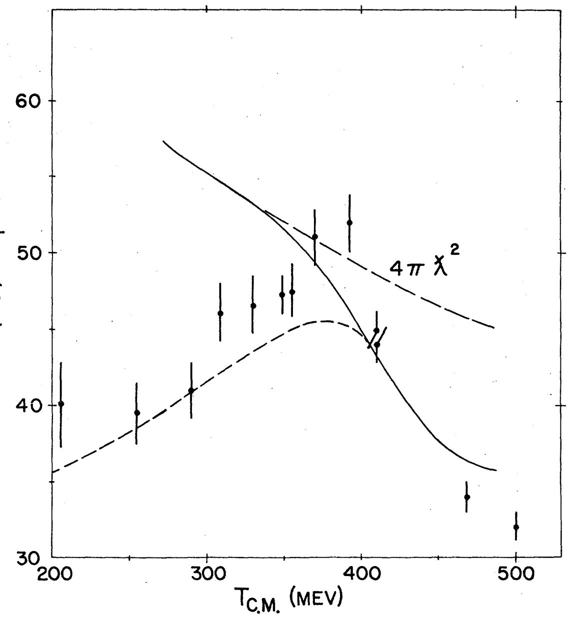

FIG. 4. The K^--p total cross section.[7] The solid line is the calculated unitarity limit for elastic scattering in the $D_{3/2}$ state, plus a constant background of 30 mb. The dashed line is a "fit" to the data, obtained by a suitable continuation of the calculated inelastic cross section.

ratio of 9/1. This agrees with the fact that no large peak is seen in the K^--n cross section. Our model implies that there should also be a peak in the K^+-n cross section, but it is impossible to predict its height from the present crude calculation. It is probably considerably smaller than the K^--p peak, because there are more inelastic channels open in the latter process, which may help to raise $\sigma_I(D_{3/2})$ close to the unitarity limit.

Additional resonances are probably implied in π-Y and K-Y systems, although detailed calculations must be carried out, which will depend upon assumptions about strange-particle couplings. The positions of these resonances will be around center-of-mass total energy $m_{\pi-\pi}+m_Y$ and $m_{K^*}+m_Y$, respectively.

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[9]We are indebted to Dr. Kenneth R. Greider for helping us to find this solution. The same solution, generalized to include a left cut, has also been found by Marcel Froissart (to be published). We are indebted to Dr. Froissart for advance communication of his results.

[10]Another method of solution is to set $A=N/D$, but giving N, in addition to the usual left cut, another cut beginning at the inelastic threshold with discontinuity $\mathrm{Im}N=(1-\eta)\,\mathrm{Re}D/2k$. The imaginary part of D is then $\mathrm{Im}D=-2k\,\mathrm{Re}N/(1+\eta)$. The resulting integral equations have been solved numerically, and it has been verified that the results coincide with Eq. (3).

[11]We continue to use the conventional term "resonance" for lack of a word more descriptive of the mechanism we propose.

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