QUANTIZATION OF FLUX IN A SUPERCONDUCTING CYLINDER

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Recent experiments of Deaver and Fairbank[1] and of Doll and Näbauer[2] have shown that the flux in a superconducting cylindrical tube is quantized. That the flux should be quantized in units of hc/e was originally suggested by London[3]; the experiments indicate units of about half this amount. Onsager[4] predicted such a result on the basis that an effective charge $e^*=-2e$, representing that of a pair, should be used. We shall show that in tubes of very small diameter and with wall thickness of the order of the penetration depth the unit may depend on dimensions and be somewhat smaller than $hc/2e$.

The theory has been discussed by Byers and Yang,[5] who showed that it is essential to take into account the statistical distribution of quasi-particle excitations, and that one could expect a flux quantum of $hc/2e$ on the basis of the microscopic theory. They considered, as we shall do here, states with current flow which exist with nonintegral values of flux.

We shall use the Gor'kov version[6] of the Ginzburg-Landau (G-L) theory[7] to discuss the effect. By use of the method of "thermal" Green's functions, Gor'kov showed that the G-L equations, originally given on a phenomenological basis, may be derived from the microscopic theory provided that the temperature is sufficiently close to T_c so that the local London equations are valid for the electrodynamics. The only difference from the original version is that an effective charge $e^*=-2e$, evidently that of a pair, appears in place of $-e$. Since in the experiments, the flux is frozen in as the specimen is cooled below T_c, the theory is valid in the significant temperature range.

Gor'kov showed that the effective wave function, $\Psi(\vec{r})$, of the G-L theory may be taken to be proportional to the local value of the energy gap. We shall take a different normalization for Ψ than that used by Gor'kov, one that is closer to the original G-L theory. We shall define it so that Ψ_0, the equilibrium value of Ψ, is such that $|\Psi_0|^2=\rho_s/\rho$, where ρ_s is the density of superconducting electrons of the two-fluid model and $\rho=Nm$, the total density. The Schrödinger-like equation for Ψ may then be written

$$-\frac{h^2}{8\pi^2 m^*}\left(\frac{\partial}{\partial\vec{r}}-\frac{2\pi i e^*}{hc}A(\vec{r})\right)^2\Psi-\alpha\Psi+\beta|\Psi|^2\Psi=0, \quad (1)$$

where $m^*=2m$ and $e^*=-2e$, representing the charge and mass of a bound pair. This equation then corresponds to that of the center-of-mass motion of the pair. The free-energy difference between superconducting and normal states is given by integrating over space and multiplying by $N/2$, the number of pairs at $T=0°K$:

$$F_s-F_n=\frac{N}{2}\int\left\{-\frac{h^2}{8\pi^2 m^*}\Psi^*\left(\frac{\partial}{\partial\vec{r}}-\frac{2\pi i e^*}{hc}A(\vec{r})\right)^2\Psi-\alpha|\Psi|^2+\tfrac{1}{2}\beta|\Psi|^4\right\}d\tau. \quad (2)$$

The density of mass flow is

$$J=\frac{N}{2}\left\{-\frac{ih}{4\pi}\left[\Psi^*\frac{\partial\Psi}{\partial\vec{r}}-\Psi\frac{\partial\Psi^*}{\partial\vec{r}}\right]-\frac{e^*}{c}|\Psi|^2A(\vec{r})\right\}. \quad (3)$$

In the absence of fields and currents, the free-energy minimum, for $|\Psi_0|^2=\alpha/\beta=\rho_s/\rho$, is $F_s-F_n=-H_c^2/8\pi=-N\alpha^2/(4\beta)$, where $H_c(T)$ is the critical field for bulk material.

Gor'kov derived the values of α and β from the microscopic theory.[8] They can be expressed in terms of H_c and the penetration depth, $\lambda(T)$. Since $\rho_s/\rho=\lambda_L^2/\lambda^2$, where $\lambda_L^2=mc^2/(4\pi e^2N)^{1/2}$ is the London penetration depth, we have

$$\alpha=H_c^2\lambda^2/(2\pi N\lambda_L^2); \quad \beta=H_c^2\lambda^4/(2\pi N\lambda_L^4). \quad (4)$$

These apply whether or not elastic scattering by impurities or other imperfections is important provided that the appropriate value for the penetration depth is used. Limiting expressions[8,9] valid near T_c are

$$\lambda_L^2/\lambda^2=2(1-t), \quad l\gg\xi_0; \quad (5)$$

$$\lambda_L^2/\lambda^2=2.73(1-t)(l/\xi_0), \quad l\ll\xi_0; \quad (6)$$

where $t=T/T_c$, l is the mean free path, and ξ_0 is the coherence distance. The limit $l\ll\xi_0$ applies to thin films.

In the absence of a magnetic field, a wave function $\Psi=\Psi_0e^{i\vec{q}\cdot\vec{r}}$ corresponds to a momentum $m^*v_s=hq/2\pi$ where v_s is the velocity of the center of mass. The net current is $\rho_s v_s$ and the increase in free energy $\frac{1}{2}\rho_s v_s^2$, in accordance with the two-fluid model.[10] A superconductor is characterized by a value of ρ_s different from zero.

To discuss the quantization of flux, we shall suppose that the specimen is in the form of a thin film of thickness $d \lesssim 2\lambda$ on the surface of a cylindrical tube of radius r and circumference $L = 2\pi r$. The current density will then be substantially uniform across the film. One may simplify the mathematics by taking x to be a linear coordinate measuring distance around the circumference and require periodic boundary conditions, $\Psi(x+L) = \Psi(x)$. If Φ is the flux through the tube, the vector potential A_x is a constant equal to Φ/L.

An appropriate periodic solution of the G-L equation is[3]

$$\Psi = \exp[2\pi i n x/L]\Psi_0, \tag{7}$$

where n is an integer. For an arbitrary flux Φ one may choose n so that the residual wave vector

$$q = (2\pi/L)(n - e^*\Phi/hc) \tag{8}$$

is in the interval $-\pi/L < q < \pi/L$. The velocity $v_s = hq/(2\pi m^*)$ and the free-energy increase will then be as small as possible.

The electric current density, $I = e^*\rho_s v_s/m^*$, is quite large even for small values of q. A value $q = \pi/L$, with $L \sim 1$ cm, gives $v_s \sim 1$ cm/sec and a current density $I \sim 10^3\ \rho_s/\rho$ amp/cm^2 in a typical metal. Generally a much smaller current density is required to produce a unit of flux, so that it is favorable to minimize the free energy by reducing q and thus v_s to nearly zero. There will then be very nearly an integral number of flux units in the cylinder. Values of q corresponding to odd multiples of π/L would also give zero current (with a different pairing condition), but with a free-energy maximum rather than a minimum.[5]

Appreciable values of q which would give nonintegral units of flux can be expected only if the thickness of the film and the radius of the cylinder are very small. When the external field is removed, the flux is produced entirely by the current flowing in the film, so that

$$\Phi = 4\pi^2 r^2 I d/c. \tag{9}$$

Setting $I = e^*\rho_s hq/(8\pi m^2)$, we may solve for q and find

$$q = 2\lambda^2 e^*\Phi/(r^2 dhc), \tag{10}$$

where $\lambda^2 = m^2c^2/(4\pi e^2\rho_s)$. Substituting in (8) and solving for Φ, we find

$$\Phi = (nhc/2e)(1 + 2\lambda^2/rd)^{-1}. \tag{11}$$

Thus we find that for tubes of very small radius, the flux quantum may be appreciably less than $hc/2e$. It is possible that the value of $0.4\,hc/e$ observed by Doll and Näbauer may be accounted for in this way. The lead tube which they used had a diameter of only 0.01 mm and λ may have been abnormally large because of scattering in the film. It should be noted that λ and thus the unit of flux may change with temperature.

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[1]B. S. Deaver, Jr., and W. M. Fairbank, Phys. Rev. Letters 7, 43 (1961).

[2]R. Doll and M. Näbauer, Phys. Rev. Letters 7, 51 (1961).

[3]F. London, Superfluids (John Wiley & Sons, New York, 1950), p. 152.

[4]L. Onsager (private communication); Phys. Rev. Letters 7, 50 (1961).

[5]N. Byers and C. N. Yang, Phys. Rev. Letters 7, 46 (1961).

[6]L. P. Gor'kov, J. Exptl. Theoret. Phys. (U.S.S.R.) 36, 1918 (1959); 37, 833 (1960) [translations: Soviet Phys. - JETP 9, 1364 (1959); 10, 593 (1960)].

[7]V. L. Ginzburg and L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) 20, 1064 (1950).

[8]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[9]P. B. Miller, Phys. Rev. 113, 1209 (1959).

[10]J. Bardeen, Phys. Rev. Letters 1, 399 (1959). It is mainly the two-fluid model, valid for all temperatures, which is required for the present discussion.