Shift and Spin Vector: New Topological Quantum Numbers for the Hall Fluids

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We discuss some new quantum numbers (spin vector) for the Hall fluid, representing orbital spin degrees of freedom. We show that the spin vectors are quantized. In the absence of impurities, two Hall fluids with different spin vectors cannot change into each other without a phase transition and closing of the energy gap. In principle the spin vector can be measured through its coupling to the curvature of space. Our formalism may be described picturesquely as a unification of electromagnetism and "gravity" in condensed-matter physics.

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It has become clear that the order in the quantum Hall fluid (and also in the closely related chiral spin fluid and anyon fluid) is not associated with broken symmetries, but is topological in character [1]. In a series of papers [2] we have shown that the order may be characterized by a symmetric integer-valued matrix K and an integer-valued charge vector t. In this paper we introduce and discuss additional topological quantum numbers, namely, a quantity $\mathcal{S}$ which we will call the shift and a spin vector s_I.

Recall that the long-distance physics of the Hall fluid can be described by a Lagrangian [2] of interacting gauge potentials a_I^μ ($I=1,2,\ldots,\kappa$). Here $\mu=0,1,2$ describe the Lorentz indices of a $(2+1)$-dimensional space-time. The Lagrangian is the sum of several pieces

$$\mathcal{L}=\frac{1}{4\pi}(aK\epsilon\partial a+2At\epsilon\partial a+2\omega s\epsilon\partial a)+aj\,. \tag{1}$$

We use the compact notation $a\epsilon\partial\beta\equiv a_\mu\epsilon^{\mu\nu\lambda}\partial_\nu\beta_\lambda$ for any two gauge potentials a_μ and β_μ. Here K is a κ by κ symmetric integer-valued matrix. Thus the first term in $\mathcal{L}$ is the by now well-known Chern-Simons interaction $\sum_{IJ}a_I^\mu K_{IJ}\epsilon_{\mu\nu\lambda}\partial^\nu a_J^\lambda$. In the fourth term the gauge potential a_I^μ couples to j_μ^I, the current of the quasiparticles (or vortices) of type I. The second term in $\mathcal{L}$ describes the coupling of the electromagnetic current $J_\mu=\sum_I t_I\epsilon_{\mu\nu\lambda}\partial^\nu a_I^\lambda$ to an electromagnetic gauge potential A_μ. These three terms appeared in our previous discussions [2], to which we refer the reader for further details.

What is new here is the third term that describes the coupling to the curvature of the space [3]. This term is motivated by the following fact. On a closed topological 2-manifold, such as a sphere [4], over which the Hall fluid is defined, the number of electrons N_e and the number of magnetic flux quanta N_ϕ going through the manifold are not simply related by $\nu N_\phi=N_e$ (where ν is by definition the filling factor) as is the case on the plane. Rather there is a shift $\mathcal{S}$ in the relation between N_e and N_ϕ:

$$N_\phi=\nu^{-1}N_e-\mathcal{S}\,. \tag{2}$$

The shift $\mathcal{S}$ is a topological quantum number that becomes numerically insignificant in the thermodynamic limit $N_e,N_\phi\to\infty$. On a torus the shift $\mathcal{S}$ always vanishes. This suggests that the quantum Hall (QH) fluid state couples to the curvature of the space in a special way to produce different shifts on different curved spaces. In order to include such a coupling, we recognize that on a curved 2-manifold over which our theory is defined there is another 1-form besides the electromagnetic gauge potential, namely, the connection ω, to which we can couple a conserved current. The third term in (1) is such a term that couples ω to the spatial components of the current $(2\pi)^{-1}s_I\epsilon_{\mu\nu\lambda}\partial_\nu a_{I\lambda}$. (We remind the reader that the connection 1-form ω is defined by the equation of parallel transport $de^a+\omega\epsilon^{ab}e^b=0$, where e^a denotes the *zweibein* on the manifold and ϵ^{ab} the antisymmetric symbol. For the sphere, for example, $e^1=d\theta$, $e^2=\sin\theta\,d\varphi$, and thus $\omega=-\cos\theta\,d\varphi$. The Riemann curvature R is given in terms of the connection ω by $R=d\omega$ in complete analogy to how the Maxwell field strength F is given in terms of the gauge potential A by $F=dA$.)

We know that moving a spinning particle along a loop in a curved space produces a phase $s\oint\omega$ in analogy to the Aharonov-Bohm phase of a charged particle moving in magnetic field $t\oint A$. Here t is the charge and s is the angular momentum of the particle. We will call s the orbital spin or simply the spin of the particle. [One should not confuse this spin with the electron spin. In this paper, the ordinary spin of the electron will be explicitly referred to as electron spin. In the FQH (fractional QH) problem, electron spin acts merely as a label, a flavor quantum number, so to speak.] We clearly see the correspondence $s\leftrightarrow t$ and $\omega\leftrightarrow A$. Thus, the covariant derivative of a field is given by $\partial_i-i(tA_i+s\omega_i)$, where t and s are the charge and spin associated with the field. Since the rotation group on a 2-manifold $O(2)=U(1)$ is Abelian, s is simply a real number. An important message that we obtained from the above discussion is that the connection ω couples to the density and the current of the spin, just like the electromagnetic potential A couples to the density and the current of the charge. In the FQH states there

are a number of condensates and each contributes to the total spin current. Out of the gauge potentials α_I^μ we can construct the spin current $J_\mu^s=(2\pi)^{-1}\sum_I s_I\epsilon_{\mu\nu\lambda}\partial^\nu\alpha_I^\lambda$, where s_I is the spin carried by a particle in the Ith condensate. We will call s_I the spin vector. This discussion accounts for the third term in (1).

To summarize, in (1) the first term describes the global or topological dynamics of the Hall fluid. The second and third terms may be thought of as describing the coupling of this dynamics to the electromagnetic field and the curvature of the space, respectively. The reason for introducing curvature is conceptual: By studying the system's response to changes in ω, we learn about the spin properties of the local system, in the same way that we learn about the charge properties of the system by studying its response to local changes in A. We thus obtain a unifying description of charge and spin. Finally the fourth term describes the coupling to quasiparticles (or vortices), in some sense the high-energy excitations in the fluid.

The physics of the Hall fluid may be read off by integrating out the gauge potentials α. For simplicity let us begin by suppressing the quasiparticle currents j. We obtain the effective Lagrangian

$$\mathcal{L}=\frac{1}{4\pi}(tA+s\omega)K^{-1}\epsilon\partial(tA+s\omega)\,. \tag{3}$$

The electromagnetic current is given by

$$J=\frac{\delta\mathcal{L}_{\rm eff}}{\delta A}=\frac{1}{2\pi}tK^{-1}\epsilon\partial(tA+s\omega)\,, \tag{4}$$

and the spin current by

$$J_s=\frac{\delta\mathcal{L}_{\rm eff}}{\delta\omega}=\frac{1}{2\pi}sK^{-1}\epsilon\partial(tA+s\omega)\,. \tag{5}$$

We integrate the time component of these two equations over the manifold and express the results as a matrix equation

$$\begin{pmatrix}N_e\\N_s\end{pmatrix}=\begin{pmatrix}tK^{-1}t & tK^{-1}s\\ sK^{-1}t & sK^{-1}s\end{pmatrix}\begin{pmatrix}N_\phi\\N_R\end{pmatrix}. \tag{6}$$

Here $N_e=\int d^2x\,J_0$ denotes the number of electrons (taken by definition to have charge 1), $N_s=\int d^2x\,J_0^s$ the number of "spin quanta," $N_\phi=(2\pi)^{-1}\int dA=(2\pi)^{-1}\int F$ the number of flux quanta, and $N_R=(2\pi)^{-1}\int d\omega=(2\pi)^{-1}\int R$ the number of curvature quanta. [By the Gauss-Bonnet theorem $N_R=2(1-g)$, where g is the genus of the manifold. For the sphere $N_R=2$.] Of the entries in the 2×2 matrix in (6) we have encountered previously $(tK^{-1}t)$, which we identified as the filling factor on the Hall conductance. The other two entries $(tK^{-1}s)$ and $(sK^{-1}s)$ are new to this paper. We see that

$$N_\phi=\frac{1}{tK^{-1}t}N_e-\frac{tK^{-1}s}{tK^{-1}t}N_R\,. \tag{7}$$

Comparing with (2) we recover our earlier result $\nu=(tK^{-1}t)$ and determine the shift to be

$$\mathcal{S}=(tK^{-1}s)\nu^{-1}N_R\,. \tag{8}$$

The shift depends on the genus, but not the metric, of the manifold, and is thus a topological quantity.

Notice the power of this effective Lagrangian approach. The dependence of the shift $\mathcal{S}$ on the total curvature (or genus) is manifest in (8) without our ever having to write down the metric explicitly. In particular, for a torus, $\mathcal{S}=0$.

Now we would like to apply the effective theory (1) to some simple QH states and determine the spin vector s_I for these states. On a sphere the shift may be worked out explicitly for noninteracting electrons [4]. For $N_\phi=2G$ (where $G=$half-integer or integer may be identified as the strength of the Dirac magnetic monopole at the center of the sphere) the single electron energy is given by $E=\frac{1}{2}\hbar\omega_c[l(l+1)-G^2]/G$ with the Landau levels corresponding to $l=G,G+1,G+2,\ldots$. The degeneracy of the lth level is $2l+1$. Thus, if L Landau levels are filled with noninteracting electrons we have $N_e=\sum_{k=0}^{L-1}[2\times(G+k)+1]=LN_\phi+L^2$ and so $\mathcal{S}=\nu=L$. The Hall fluid with L Landau levels filled is described by K equal to the $L\times L$ identity matrix and by $t_I=1$, $I=1$ to L. The Ith Landau level contributes to the shift $\nu^{-1}(2I-1)$. Referring to (8) we see that the spin of the noninteracting electrons in the Ith Landau level is given by

$$s_I=(I-\tfrac{1}{2})\,. \tag{9}$$

Thus the spins are given by $\frac{1}{2},\frac{3}{2},\ldots$. One can easily check that such a choice of the spin gives the right value of the shift. Intuitively, we may think of s as effective angular momentum carried by the electrons due to the cyclotron motion. In a semiclassical picture, we may think of electrons in different Landau levels as moving around in Larmor orbits of different sizes and hence of different angular momenta. This, we believe, is the physics behind (9). The nontrivial spin vectors and consequent coupling to the curvature are due to the cyclotron motion.

The spin vector in the IQH (integer QH) states is quantized according to Dirac's argument. Indeed, we see that for a sphere the connection $\omega=-\cos\theta\,d\varphi$ is identical to the gauge potential A corresponding to a monopole of twice the Dirac unit. (This is again the statement that $N_R=2$ for the sphere.) A particle carrying spin s going around a closed loop C will acquire a quantum phase $\exp(is\oint_C\omega)$. To repeat the standard argument, to cover the northern hemisphere we have to use $\omega_N=(1-\cos\theta)\times d\varphi$, and the southern hemisphere, $\omega_S=(-1-\cos\theta)d\varphi$. Requiring that the phase factor acquired by a particle traveling around the equator be equal when calculated with other ω_N or ω_S, we obtain that s has to be a half-integer or an integer.

Because of this quantization we expect that even though the result in (9) was derived for noninteracting electrons it should continue to hold in the presence of in-

teractions, as long as the interactions are not so drastic as to close the gaps.

For a FQH state with its strong correlations, we can read off the shift if we know the wave function explicitly. In Ref. [2] we derived the result that for a multilayered wave function described by K in the symmetric basis (with $t_I=1$, all I): $\prod_{I,J=1}^{\kappa}\prod_{i<j}(z_i^{(I)}-z_j^{(J)})^{K_{IJ}}$, the shift on the sphere is given by

$$\mathcal{S}=\frac{1}{\nu}\sum_{IJ}(K^{-1})_{IJ}K_{JJ}. \tag{10}$$

Referring to the general formula (8) we see that the spin vector is given by

$$s_I=\tfrac{1}{2}K_{II}. \tag{11}$$

For the hierarchical states described by K^h in the hierarchical basis (with $t_I=\delta_{I1}$) we have

$$\mathcal{S}=\frac{1}{\nu}\sum_{I}[(K^h)^{-1}]_{1I}K^h_{II} \tag{12}$$

and thus

$$s_I=\tfrac{1}{2}K^h_{I1}. \tag{13}$$

(The results for $\mathcal{S}$ are quoted for the sphere.) For instance, for the hierarchical state characterized by

$$K^h=\begin{pmatrix} p_1 & l \\ l & p_2 \end{pmatrix}$$

and filling factor $\nu=(p_2-l^2/p_1)^{-1}$ we have $\mathcal{S}=p_1-l$. [The $\nu=\frac{2}{5}$ state is described by $K^h=\left(\begin{smallmatrix}3 & -1\\ -1 & 2\end{smallmatrix}\right)^{-1}$ and $\mathcal{S}=4$. The $\nu=\frac{2}{7}$ state is described by $K^h=\left(\begin{smallmatrix}3 & 1\\ 1 & -2\end{smallmatrix}\right)^{-1}$ and $\mathcal{S}=2$.] From the above we see that the spin vector in the (generalized) hierarchical FQH states may be quite complicated. However, for the Laughlin states with filling fraction $1/m$ we have a simple result $s=m/2$ and $\mathcal{S}=m$.

In Ref. [2], we construct K by an iteration procedure. In what was called step A there, two given matrices K_1 and K_2 are combined into

$$K=\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}.$$

It follows trivially that the spin vectors simply combine into $\left(\begin{smallmatrix}s_1\\ s_2\end{smallmatrix}\right)$ and the shift $\mathcal{S}=(\nu_1+\nu_2)^{-1}(\nu_1\mathcal{S}_1+\nu_2\mathcal{S}_2)$. In step B, we let $K\rightarrow K+C$, where C is defined to be a matrix in which every entry is equal to 1. We find in the symmetric basis that the shift $\mathcal{S}\rightarrow\mathcal{S}+N_R$ and $s\rightarrow s+t/2$. Using these two iteration rules we can easily work out the shift and the spin vectors for any state generated by the iterations. We can use these iteration steps to construct hierarchy states with the same matrix K as in the standard hierarchy states [2]. However, the states obtained by iteration and the standard hierarchy states may have different shift and spin vectors. The ambiguity arises because in step A of the construction, with $K_1=1$, the associated spin s_1 may or may not be taken to be $\frac{1}{2}$, depending on whether $K_1=1$ is associated with the lowest or a higher Landau level.

Let us now restore the quasiparticles in (3). The prescription, as we can see from (2), is to simply replace $tA+s\omega$ in (3) by $tA+s\omega+(\epsilon\partial)^{-1}2\pi j$. The resulting effective Lagrangian would then have a jj term which, as was explained in Ref. [2], tells us about the statistics of the quasiparticles, a jA term, which tells us about the charge of the quasiparticles, and finally a $j\omega$ term, which is new to this paper. This new term reads

$$\mathcal{L}_{\omega j}=\omega sK^{-1}j \tag{14}$$

and tells us that the spin of the quasiparticle of type I is given by $\sum_J K^{-1}_{IJ}s_J$. For a generic excitation that contains l_I-type I quasiparticles, the total spin, charge, and statistics are given by

$$S=\sum_{IJ}l_I(K^{-1})_{IJ}s_J,\quad Q=\sum_{IJ}l_I(K^{-1})_{IJ}t_J,$$
$$\frac{\theta}{\pi}=\sum l_IK^{-1}_{IJ}l_J. \tag{15}$$

Note that in general the spin statistics theorem does not hold.

From Ref. [2], we know that there are κ types of electron excitations in the Hall fluid, where κ is the rank of K. The electron excitation of the Ith type is specified by (in the symmetric basis)

$$l_J^{(I)}=K_{IJ},\quad I=1,\ldots,\kappa. \tag{16}$$

Inserting this into the first equation in (15) we obtain that the spin of the Ith-type electron is given by

$$S_e^{(I)}=s_I. \tag{17}$$

The second equation in (15) tells us that the charge of the electron is $Q_e^{(I)}=t_I$, that is, equal to 1 in the symmetric basis.

We can derive from (17) a quantization condition on the spin vector in general. The effective theory (1) as written is self-consistent for any real spin vector s_I. However, such an effective theory may not describe an electron system. In order for the effective theory to describe an electron system, we require the presence of κ types of electron excitations as suggested in Ref. [2]. In particular, we require that one can always add an electron to the Hall fluid on the sphere. If we add an I-th type electron on the sphere, such an electron will see effectively $Q_e^{(I)}N_\phi+S_e^{(I)}N_R$ number of flux quanta passing through the sphere. Dirac's topological quantization requires this number to be an integer. Because $Q_e^{(I)}N_\phi$ is already an integer and $N_R=2$, we find that $S_e^{(I)}=s_I$ must be a multiple of $\frac{1}{2}$ (in the symmetric basis). Generic FQH states (including multilayer and electron-spin unpolarized FQH states) are classified by the K matrix, the charge vector t, and the spin vector s.

The above result and the result in Ref. [2] can be summarized as follows. We can always redefine the fields in (1) and choose the symmetric basis $t_I=1$. In order for the effective theory (1) to describe an electron system, K must be a symmetric integer matrix with odd diagonal elements and the spin vector must be $\frac{1}{2}$ times an integer

vector.

The spin vector s_I, while topological, is not as robust as the charge vector t_I and the matrix K, since rotational symmetry may be readily broken by impurities. Electrons scattering on impurities may change their effective orbital angular momenta (for example, by jumping from one Landau level to another). Thus the above results about the spin vector are correct only for pure systems without impurities.

From (6) we see also that $N_s=(sK^{-1}t)N_\phi+(sK^{-1}s)\times N_R$. The quantity that appears in the shift also controls the relationship between N_s and N_ϕ, while $\nu_s\equiv sK^{-1}s$ may be thought of as a "spin filling factor."

Now we would like to use the concept of the spin vector to address a physical question. We know there are two $\nu=\frac{2}{5}$ FQH states. One is the electron-spin polarized hierarchical state. The second has unpolarized electron spin and is described by wave function of type (3,3,2) [5]. Because the magnetic field breaks the electron-spin rotational symmetry, both states have the same electron-spin symmetry. The two states are also described by the same matrix $K=\left(\begin{smallmatrix}3&2\\2&3\end{smallmatrix}\right)$ (in the symmetric basis). We would like to ask whether the two states belong to the same universality class or not. In other words, can we deform the electron Hamiltonian so that one state continuously change into the other? We notice that the two states have different spin vectors: $(s_1,s_2)=(\frac{3}{2},\frac{5}{2})$ for the hierarchical state and $(s_1,s_2)=(\frac{3}{2},\frac{3}{2})$ for the (3,3,2) state (in the symmetric basis). In the absence of impurities the spin vector is quantized and cannot change continuously. Therefore, for a pure system the two quantum Hall states belong to two different universality classes and cannot change into each other continuously. However, in the presence of impurities, a continuous crossover may occur.

The discussion on edge excitations given in Ref. [6] may be formally generalized. The effective $\mathcal{L}=(4\pi)^{-1}\times\mathcal{A}\mathcal{N}\epsilon\partial\mathcal{A}$ in (3), where we have introduced the notation $\mathcal{A}\equiv(A,\omega)$ and $\mathcal{N}\equiv$ the 2×2 matrix in (6), is not strictly gauge invariant. If our topological 2-manifold $\Omega=\Omega_1+\Omega_2$ is divided into two regions Ω_1 and Ω_2 sharing a common boundary $\partial\Omega_1=-\partial\Omega_2$, and in which $\mathcal{N}$ equals $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively under the gauge transformation $\mathcal{A}\rightarrow\mathcal{A}+\partial\Lambda$ the action $\mathcal{S}=\int_\Omega\mathcal{L}$ varies by $(4\pi)^{-1}\times\int\Lambda\Delta\mathcal{N}\epsilon\partial\mathcal{A}$ with $\Delta\mathcal{N}\equiv\mathcal{N}_1-\mathcal{N}_2$. (The special case in which there is no Hall fluid in Ω_2 is described by setting $\mathcal{N}_2$ to zero.) Thus, the action $\mathcal{S}$ must be supplemented by an action defined on the boundary

$$\mathcal{S}_B=\int dt\,dx\,dt'dx'\tfrac{1}{2}\mathcal{A}_\mu(t,x)\Pi^{\mu\nu}(t-t',x-x')\mathcal{A}_\nu(t',x').$$

(Here $\mu,\nu=0,1$.) The requirement that the total action $\mathcal{S}+\mathcal{S}_B$ must be gauge invariant implies that $k^\mu\Pi_{\mu\nu}=(\Delta\mathcal{N}/4\pi)k^\mu\epsilon_{\mu\nu}$. As shown in Ref. [6], there must be gapless edge excitations. For instance, $\Pi_{00}=-k\times\sum_A\eta_A/(\omega-c_Ak)$. The only difference is that here η_A are 2×2 matrices, not just real numbers, satisfying $\sum_A\eta_A=\Delta\mathcal{N}/2\pi$. Positivity requires that the diagonal elements of η_A must be positive if c_A is negative, and negative if c_A is positive. A particularly interesting case occurs when the two fluids in Ω_1 and Ω_2 have the same K and t (and thus in particular the same filling factor) but different s. From (6) we see that the 11 entry in $\Delta\mathcal{N}$ vanishes but not the other entries. From this we deduce that if there are massless excitations at all they cannot all move in the same direction.

This discussion, however, is entirely formal since the microscopic physics responsible for the existence of the edge would presumably also break local rotation invariance and general covariance. Therefore a nonzero 22 entry in $\Delta\mathcal{N}$ may not imply the existence of additional edge states. In contrast, electromagnetic gauge invariance is absolute and respected by any microphysics.

We conclude with the cautionary remark that in (1) we are dealing with curved space, not curved spacetime. (In other words, of the connection $\omega_\mu^{a\beta}$ in curved spacetime $\mu=0,1,2$, $\alpha,\beta=0,1,2$, only the components $\omega_i^{ab}=\omega_i\epsilon^{ab}$ with $i=1,2$, $a,b=1,2$ are nonzero.) After all, the microscopic physics here is not even Lorentz invariant, let alone general covariant. The effective $\mathcal{L}$, however, may be promoted to be general covariant, with ω_μ^{ab} nonzero. We remark in passing that we can imagine curved space appearing in condensed-matter physics if the electron hopping is such that the mass matrix in the term $\int d^2x\,\frac{1}{2}(m^{-1})_{ij}(\partial_i+iA_i)\psi^\dagger(\partial_i-iA_i)\psi$ in the Hamiltonian depends on position. The $(m^{-1})_{ij}$ may be identified with $\sqrt{g}g^{ij}(x)$. However, we may interpret this as an electron moving on a curved space only if g ($\equiv\det g_{\mu\nu}=\det g_{ij}$) is a constant since we do not see how the requisite factor of $\sqrt{g}$ could appear in the kinetic energy term in the action $\int d^2x\,dt\sqrt{g}\psi^\dagger i\partial_0\psi$.

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