Exact Mass Gap of the Chiral SU(n) × SU(n) Model

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Using the Bethe ansatz equation the mass gap of the chiral SU(n)×SU(n) model is calculated in terms of the Λ parameter in the modified minimal subtraction scheme. The result is in reasonable agreement with recent Monte Carlo data.

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In a recent paper Hasenbusch and Meyer [1] have presented high-quality data on the mass gap of the SU(3)×SU(3) chiral sigma model and of the CP(3) model in two dimensions regularized on the lattice. Their motivation for this study was to test the efficiency of the multigrid Monte Carlo algorithm to overcome critical slowing down in models for which efficient cluster algorithms have so far not been devised.

The SU(n)×SU(n) and CP(n) models have extremely interesting properties akin to those of non-Abelian gauge theories. The chiral model has in the framework of Migdal recursion relations [2] the same renormalization-group trajectories as pure gauge theories in four dimensions; and the CP(n) models have instanton solutions. Both are asymptotically free and thought to have a dynamically generated mass gap.

Both models have an infinite number of conservation laws at the classical level. These survive quantization for the chiral model but are spoiled by quantum anomalies in the CP(n) model [3]. The presence of an infinite number of conservation laws, implying the absence of particle production, enables the determination of the S matrix up to so-called Castillejo-Dalitz-Dyson (CDD) factors [4]. If one fixes these and makes some extra mild assumptions one can then determine the ratio of the mass gap to the Λ parameter in these class of models as was first done by Hasenfratz, Maggiore, and Niedermayer [8,9] for the O(N) sigma model. In this paper we extend this computation for the principal chiral model and compare the results with the data of Ref. [1] referred to above.

The action of the SU(n)×SU(n) chiral model is

$$S=\frac{1}{g^2}\int dx\,\mathrm{tr}[\partial_\mu U(x)\,\partial_\mu U^\dagger(x)]\,, \tag{1}$$

where $U\in \mathrm{SU}(n)$. The action is invariant under both left- and right-handed global SU(n) transformations $U(x)\to V_L U(x)$ and $U(x)\to U(x)V_R^T$ with corresponding conserved Noether currents

$$\begin{aligned} J_\mu^L(x)&=U(x)\,\partial_\mu U^\dagger(x)\,,\\ J_\mu^R(x)&=U^*(x)\,\partial_\mu U^T(x)\,. \end{aligned} \tag{2}$$

The lowest massive particle states are thought to belong to the fundamental representation (n,n) and to the conjugate representation (their antiparticles). The S matrix has been independently proposed by Wiegmann [10] and by Abdalla, Abdalla, and Lima-Santos [11]. It has SU(n)×SU(n) symmetry and the scattering of two particles in the fundamental representation is described by

$$\begin{aligned} &\langle p_1',\alpha_1',\beta_1',p_2',\alpha_2',\beta_2',\mathrm{out}\,|p_1,\alpha_1,\beta_1,p_2,\alpha_2,\beta_2,\mathrm{in}\rangle\\ &\quad=\delta(p_1-p_1')\delta(p_2-p_2')S(\theta)\left[\mathcal{P}^+ +\frac{x+i\Delta}{x-i\Delta}\mathcal{P}^-\right]_{\alpha_1,\alpha_2}^{\alpha_1',\alpha_2'}\left[\mathcal{P}^+ +\frac{x+i\Delta}{x-i\Delta}\mathcal{P}^-\right]_{\beta_1,\beta_2}^{\beta_1',\beta_2'}\\ &\qquad+(\alpha_1\leftrightarrow\alpha_2,\beta_1\leftrightarrow\beta_2,p_1\leftrightarrow p_2)\,. \end{aligned} \tag{3}$$

Here the momenta are expressed in terms of their mass m and rapidity according to $p_i=m(\cosh\theta_i,\sinh\theta_i)$; and $\theta=\theta_1-\theta_2$ is the relative rapidity. Furthermore $\mathcal{P}^+$ and $\mathcal{P}^-$ are projections onto the symmetric and the antisymmetric channels, respectively, and

$$x=\theta/2\pi,\quad \Delta=1/n\,. \tag{4}$$

The particle-antiparticle S matrix, which has the property of absence of reflection, will not be needed here. The amplitude $S(\theta)$ in the completely symmetric channel is given by

$$S(\theta)=-\frac{\Gamma^2(1+ix)\Gamma(\Delta-ix)\Gamma(1-\Delta-ix)}{\Gamma^2(1-ix)\Gamma(\Delta+ix)\Gamma(1-\Delta+ix)}\,. \tag{5}$$

The full S matrix can be considered as the direct product of chiral Gross-Neveu S matrices times a CDD factor [10]

$$X(\theta)=\frac{\sinh\pi(x+i\alpha)}{\sinh\pi(x-i\alpha)}\,, \tag{6}$$

with $\alpha=\Delta$. This factor leads to a pole on the physical sheet in the antisymmetric channel, which through "fusion" induces a whole tower of bound states with masses

$$m_r=m\frac{\sin r\pi\Delta}{\sin\pi\Delta},\quad 1\le r\le n-1\,. \tag{7}$$

The bound states of r particles transform as totally antisymmetric tensors of rank r. Note that states with $r=k>n/2$ have the same mass as states with $r=n-k$, which corresponds to the picture [12] that antiparticles of the kth state can be considered as bound states of $n-k$ particles. The presence of bound states is consistent with the picture that the model is equivalent to a model of fermions whose interaction is attractive [6].

Finally we note the fascinating property, stressed in Ref. [11], that the S matrix has a simple expansion in powers of $1/n$. On the other hand, although a systematic expansion of matrix models is in principle given by the topological expansion [13], even the lowest order, the planar graphs, has not been solved explicitly.

The procedure now is to calculate the free energy in the presence of a large chemical potential h coupled to a Noether charge Q [8] of the $SU(n)\times SU(n)$ symmetry. The charge should be chosen appropriately, the optimal choice being that the coupling of a chemical potential to it favors the presence of just one particle with the highest value of the charge in the ground state. This is possible for the chiral model if we chose $Q=Q_L+Q_R$ with Q_L and Q_R acting on the fundamental multiplet with the same matrix, i.e., $Q_L|\alpha,\beta\rangle=q_\alpha|\alpha,\beta\rangle$, $Q_R|\alpha,\beta\rangle=q_\beta|\alpha,\beta\rangle$, with a special choice $q=\tilde{q}$:

$$\tilde{q}=\mathrm{diag}(\tfrac{1}{2},v,v,\ldots,v)\,, \tag{8}$$

where $v=-1/(2n-2)$. Note we have normalized the charges such that the largest eigenvalue of Q is 1. Only one particle of the fundamental multiplet corresponding to $r=1$ in Eq. (7) has eigenvalue $+1$ and all other states have eigenvalues strictly less than 1 (since the bound states belong to completely antisymmetric representations). Based on our experience with the $SU(n)$ chiral Gross-Neveu model [14] we expect that for this special choice of the charge the state with the minimal energy will consist *only* of the $Q=+1$ particles.

For a finite system of P identical bosons the discrete allowed momenta in a periodic box of size L are determined by the basic set of equations

$$e^{im\sinh\theta_j L}\prod_{r\neq j}S(\theta_j-\theta_r)=1,\quad j=1,2,\ldots,P\,. \tag{9}$$

Using the fact that the two-particle wave function vanishes for equal momenta [i.e., $S(0)=-1$ as holds in our case (5)], one can give convincing arguments that for theories with a factorizable S matrix this equation holds in the thermodynamic limit ($L\to\infty$, $P/L=\rho=$const) for any density [15], and thereby obtain an integral equation for the density $\rho(\theta)$.

It is then a small step to derive the integral equation for the free energy (see, e.g., Ref. [9]), which involves as its kernel $K(\theta)$ the derivative of the scattering phase shift. This equation can be solved using the generalized Wiener-Hopf method, briefly summarized in Refs. [8,16] (see also the appendix of Ref. [17]). Here we just state the result for a class of models which includes the nonlinear σ model and the principal chiral model. Writing the Fourier transform $\tilde{K}$ of the integral kernel as $1-\tilde{K}(\omega)=1/G_+(\omega)G_-(\omega)$, where $G_+(\omega)$ [$G_-(\omega)$] are analytic in the upper [lower] half plane, in these models $G_+(i\xi)$ for small values of ξ has an expansion of the form [18]

$$G_+(i\xi)=\frac{k}{\sqrt{\xi}}e^{-a\xi\ln\xi}(1-b\xi+\cdots)\,. \tag{10}$$

The dependence of the free energy density $f(h)$ on h for $h\gg m$ is, up to terms vanishing in the limit $h/m\to\infty$, given in terms of $G_+(i)$ and the parameters k, a, b appearing in (10) through

$$\delta f(h)\equiv f(h)-f(0)=-\frac{h^2}{4}k^2\left[\ln\frac{h}{m}+\ln\left(\frac{\sqrt{2\pi}ke^{-b}}{G_+(i)}\right)-1+a(\gamma_E-1+\ln 8)+(a+\tfrac{1}{2})\ln\ln\frac{h}{m}+\cdots\right]. \tag{11}$$

For the principal chiral model one gets

$$a=0,\quad b=\Delta\ln\Delta+(1-\Delta)\ln(1-\Delta)\,, \tag{12}$$

$$k=\frac{1}{\sqrt{2\pi\Delta(1-\Delta)}}\,, \tag{13}$$

and

$$\frac{ke^{-b}}{G_+(i)}=\frac{1}{\Gamma(1+\Delta)\Gamma(2-\Delta)}\,. \tag{14}$$

The calculations done for the nonlinear sigma model [8] already fix the coefficients in Eq. (11). In Ref. [8], however, some of the corresponding numbers have been obtained numerically (to a precision 10^{-8}). The computation can be simplified further by methods developed by one of us [19], and all the coefficients can be obtained analytically.

Next we turn to the perturbative computation of the free energy. The Euclidean action for a chemical potential h coupled to a charge Q is

$$S=\frac{1}{g^2}\int dx\,\mathrm{tr}[D_\mu U(x)\bar{D}_\mu U^\dagger(x)]\,, \tag{15}$$

where g is the bare coupling and the "covariant derivative" D is given by

$$\begin{aligned}D_\mu U&=\partial_\mu U-h\delta_{\mu 0}(q_L U+Uq_R^T)\,,\\ \bar{D}_\mu U^\dagger&=\partial_\mu U^\dagger+h\delta_{\mu 0}(U^\dagger q_L+q_R^T U^\dagger)\,.\end{aligned} \tag{16}$$

To perform the standard perturbative computation in the bare coupling, we regularize the theory using dimensional regularization. The computation just involves a one-loop calculation which parallels that in the nonlinear sigma model and hence will not be repeated here. Consider now instead of $\tilde{q}$ a more general charge with Q_L

and Q_R acting on the U's with the same matrix $q=\mathrm{diag}(q_1,q_2,\ldots,q_n)$ with $\sum q_i=0$. As in Eq. (8), the largest eigenvalue is normalized to $\frac{1}{2}$: $q_1=\frac{1}{2}\geq|q_2|\geq\cdots\geq|q_n|$, but otherwise $q_2,\ldots,q_n$ are arbitrary. The free energy has a perturbative expansion in the running renormalized coupling $\bar{g}(h)$ in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme of the form

$$\delta f(h)=-\frac{4h^2}{\bar{g}^2(h)}\sum_j q_j^2-\frac{h^2}{2\pi}\sum_{i>j}(q_i-q_j)^2[\ln|q_i-q_j|-\tfrac{1}{2}]+O(\bar{g}^2). \tag{17}$$

The explicit h dependence is obtained by expressing the running coupling in terms of the Λ parameter through the relation:

$$\frac{1}{\bar{g}^2(h)}=\beta_0\ln(h/\Lambda_{\overline{\mathrm{MS}}})+\frac{\beta_1}{\beta_0}\ln\ln(h/\Lambda_{\overline{\mathrm{MS}}})+O\left(\frac{1}{\ln(h/\Lambda_{\overline{\mathrm{MS}}})}\right). \tag{18}$$

Here β_0 and β_1 are the universal one- and two-loop beta-function coefficients of the principal chiral model [20]

$$h\frac{\partial}{\partial h}\bar{g}^2(h)=-\beta_0\bar{g}^4(h)-\beta_1\bar{g}^6(h)-\cdots, \tag{19}$$

$$\beta_0=\frac{1}{8\pi\Delta},\quad \beta_1=\frac{1}{128\pi^2\Delta^2}. \tag{20}$$

By comparing the leading terms in the two expressions (11) and (17) one observes that the perturbative free energy is in general lower than that obtained from the integral equation. This follows from the relation $\sum_j q_j^2\geq q_1^2 n/(n-1)$, where the equality holds only when $q=\tilde{q}$. This means that the ground state is in general a mixture of particles. On the other hand, it supports our working hypothesis that for $q=\tilde{q}$ only the $Q=+1$ particles appear. Taking $q=\tilde{q}$ one sees that the $\ln\ln h$ terms also match since the relation

$$\frac{\beta_1}{\beta_0^2}=a+\tfrac{1}{2} \tag{21}$$

holds. We would like again to express our opinion that it is a nontrivial fact that the S matrix in conjunction with the integral equation reproduces the universal part of the perturbative beta function. This fact has been observed in all models considered so far.

Comparing the remaining terms in Eqs. (11) and (17) we obtain the desired result:

$$\frac{m}{\Lambda_{\overline{\mathrm{MS}}}}=\left(\frac{8\pi}{e}\right)^{1/2}\frac{\sin\pi\Delta}{\pi\Delta}. \tag{22}$$

We have also investigated the question how the addition of CDD factors of the form (6) modifies our result. First we note that such factors with $0<\alpha<\frac{1}{2}$ introduce extra poles in the physical strip and therefore change the spectrum of the model. In particular, we would necessarily have bound states also in the symmetric channel. If we want to keep the spectrum unchanged, we can only use CDD factors with $-\frac{1}{2}<\alpha<0$. By computing the modified kernel we find, however, that the presence of such factors changes the thermodynamics of the model drastically and the obtained free energy contradicts the perturbative result. Presence of bound states in the symmetric channel probably leads to a similar disagreement because they would have larger charge/mass ratio. Therefore we conclude that additional CDD factors are very unlikely.

The standard lattice action for the principal chiral model is [1,21]

$$S=-\tfrac{1}{2}\beta\sum_{x,\mu}\mathrm{tr}[U(x)U^\dagger(x+\hat{\mu})-1]+\mathrm{H.c.}, \tag{23}$$

where $\beta=2/g_0^2$.

The ratio of the lattice Λ parameter to that of the $\overline{\mathrm{MS}}$ scheme can be obtained in various ways, e.g., by computing the lattice two-point function and comparing it with that obtained using dimensional regularization [20]. An even simpler way is to calculate the free energy $\delta f(h)$ with lattice regularization, the correct form of the action is in this case being [22]

$$S=-\tfrac{1}{2}\beta\sum_{x,\mu}\mathrm{tr}[U(x)e^{h_\mu q}U^\dagger(x+\hat{\mu})e^{h_\mu q}+U^\dagger(x)e^{-h_\mu q}U(x+\hat{\mu})e^{-h_\mu q}-2]. \tag{24}$$

One obtains

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_L}=\sqrt{32}\exp\left(\frac{n^2-2}{2n^2}\pi\right), \tag{25}$$

which also follows from the result of Kogut and Shigemitsu [21] using $\Lambda_{\overline{\mathrm{MS}}}=\Lambda_{\mathrm{PV}}$. Note also the agreement of the results (22) and (25) for the model with $n=2$ with those of the $O(N)$ sigma model for $N=4$ as they should since the models are equivalent in this case.

The published data by Hasenbusch and Meyer [1] on the SU(3)×SU(3) model (which agreed with previous studies [23] for small correlation lengths) showed for δ_m, the ratio of the mass to the two-loop Λ parameter,

$$\delta_m=m(4\pi\Delta\beta)^{-1/2}e^{4\pi\Delta\beta}, \tag{26}$$

a steep rise to a maximum at correlation length ~ 10 and then a steady fall. Recently they [24] have extended their previous data in two ways. First they have repeated the point at $\beta=2.1$ for lattice size 512 to study possible finite-size effects. The new data give a $\sim 5\%$ larger

TABLE I. Values for $m/\Lambda_{\overline{\rm MS}}$ for various n.

n	$m/\Lambda_{\overline{\rm MS}}$	$\Lambda_{\overline{\rm MS}}/\Lambda_L$	m/Λ_L
2	1.93577	12.4071	24.0172
3	2.51463	19.1940	48.2659
4	2.73759	22.3609	61.2149
∞	3.04069	27.2122	82.7439

correlation length which brings the value of δ_m at $\beta=2.1$ slightly below that at $\beta=2.0$. Furthermore they now have a new data point at $\beta=2.2$ (where the correlation length is around 58) with a resulting value for δ_m close to that at $\beta=2.1$, viz., 57.5(10) [25]. The estimate for $m/\Lambda_{\overline{\rm MS}}$ that one would obtain without knowledge of the three-loop lattice beta-function coefficient would be about 20% off the exact value (see Table I). On the one hand, it shows again how elusive it is to get this ratio correct to a few percent. On the other hand, the degree of agreement strengthens the general assumptions based on the standard folklore on these models [26].

Finally we would like to point out that it would be nice to invest a similar elaborate lattice study to the SU(4) $\times$SU(4) model. This is one of the simplest bosonic models with conservation laws which has a bound state and therefore lends itself to precise investigations of scaling rather than asymptotic scaling.

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