Grand Unification, Gravitational Waves, and the Cosmic Microwave Background Anisotropy

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We reexamine the stochastic gravitational wave background resulting from inflation and its effect on the cosmic microwave background radiation (CMBR). Measurement by the Cosmic Background Explorer satellite of a CMBR quadrupole anisotropy places an upper limit on the vacuum energy during inflation of $\approx 5\times 10^{16}$ GeV. Gravitational waves from inflation could produce the entire observed signal if the vacuum energy during inflation was as small as 1.5×10^{16} GeV at the 95% confidence level. This coincides with recent estimates of grand unification scales inferred from renormalization-group arguments, for supersymmetric grand unified theories. Further tests of this possibility are examined.

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The observation by the differential microwave radiometers aboard the Cosmic Background Explorer (COBE) satellite of large-scale anisotropies in the cosmic microwave background radiation (CMBR) [1] is probably the most important discovery in cosmology since the discovery of the CMBR itself [2]. Such anisotropies cannot have been induced by causal processes which were initiated after the era of recombination and thus represent true primordial fluctuations resulting from physics associated with the initial conditions of the Friedman-Robertson-Walker cosmology. These initial conditions are likely to have resulted from processes associated with either an inflationary phase or new Planck-scale physics. Only in the former case can explicit predictions be made and the COBE data on the temperature correlation function are remarkably consistent with a flat Harrison-Zel'dovich spectrum as predicted from inflation. (It should be noted of course that the COBE results do not unambiguously *prove* inflation.)

Inflation predicts at least two sources of CMBR anisotropies. Scalar energy density fluctuations on the surface of last scattering induced by primordial (dark) matter density perturbations will, depending on the shape of the inflation potential, result both in subsequent structure formation and in appropriate dipole and higher moment anisotropies in the CMBR [3]. Based on the observed dipole asymmetry one can determine an upper limit on the expected quadrupole anisotropy in the case of a flat spectrum. In addition, if the scale of inflation is sufficiently high, long-wavelength gravitational waves will be generated during inflation whose reentry into the horizon can result in a large-scale observed quadrupole and higher multipole anisotropies in the CMBR today [4]. Inflation is not the only method of generating such a background of waves [5], but it is the most well motivated.

Here we reexamine gravitational wave generation during inflation, determine the predicted signal in the CMBR, and compare this with the COBE data. Our detailed estimates update and reconcile various earlier analyses. We present a likelihood function for the probability that inflation at a given scale would result in a quadrupole anisotropy at least as big as that which is observed (and also compare this to the predicted quadrupole anisotropy from scalar density perturbations). We thus place limits on the range of scales for which gravitational waves from inflation could result in all or most of the observed quadrupole anisotropy. These scales are consistent with the scale at which the $SU(3)\times SU(2)\times U(1)$ gauge couplings can be unified, based on a renormalization-group extrapolation of low-energy data, for various grand unified theory (GUT) models. We find this coincidence both suggestive and exciting, and consider other observational probes by calculating the energy density stored in a stochastic gravitational wave background today.

Since the work of Starobinsky [6], it has been recognized that a period of exponential expansion in the early Universe would lead to the production of gravitational waves. Rubakov and collaborators [7] used this to limit the scale of inflation and with it the scale of GUTs. Since that time analyses designed to more accurately compute gravitational wave backgrounds and compare limits and predictions have been developed [8–12]. More recently the limits on the quadrupole anisotropy of the microwave background had improved. It thus seemed, even before COBE, a good time to reanalyze the gravitational wave limits. Many of the analytic techniques and results we derive have appeared scattered in the literature, but we have made some effort to check, unify, and reconcile the previous methods and in the process correct any errors. Further details can be found in [13].

It is convenient to write the metric in the $k=0$ Robertson-Walker form

$$ds^2=R^2(\tau)(-d\tau^2+d\mathbf{x}^2)\,, \tag{1}$$

where $d\tau=dt/R(t)$ is the conformal time. In a universe which undergoes a period of exponential inflation, followed by a radiation dominated epoch and then a matter dominated phase, $R(\tau)$ and $\dot R(\tau)$ can be matched at the transition points, assuming that the transitions between phases are sudden (see also [14]). We define τ_1 to be the (conformal) time of radiation-matter equality, and τ_2 to be the end of inflation. The Hubble constant during inflation H and vacuum energy density V_0 driving the inflation are related by

$$H^2=\frac{8\pi}{3}\frac{V_0}{m_{\rm Pl}^2}=\frac{8\pi}{3}m_{\rm Pl}^2 v\,, \tag{2}$$

where we introduce the notation $v \equiv V_0/m_{\rm Pl}^4$.

A classical gravitational wave in the linearized theory is a ripple on the background space-time

$$g_{\mu\nu}=R^2(\tau)(\eta_{\mu\nu}+h_{\mu\nu}), \quad \eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1), \quad h_{\mu\nu}\ll 1. \tag{3}$$

In transverse traceless (TT) gauge the two independent polarization states of the wave are denoted as $+,\times$. In the linear theory the TT metric fluctuations are gauge invariant. We write

$$h_{\mu\nu}(\tau,\mathbf{x})=h_\lambda(\tau;\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\epsilon_{\mu\nu}(\mathbf{k};\lambda), \tag{4}$$

where $\epsilon_{\mu\nu}(\mathbf{k};\lambda)$ is the polarization tensor and $\lambda=+,\times$. The equation for the amplitude $h_\lambda(\tau;\mathbf{k})$ is obtained by requiring the perturbed metric (3) to satisfy Einstein's equations to $O(h)$. As was first noted by Grishchuk [15] the equation of motion for this amplitude is then identical to the massless Klein-Gordon equation for a plane wave in the background space-time. In this way, one finds each polarization state of the wave behaves as a massless, minimally coupled, real scalar field, with a normalization factor of $\sqrt{16\pi G}$ relating the two.

The spectrum of gravitational waves generated by quantum fluctuations during the inflationary period can be derived by a sequence of transformations relating creation and annihilation operators defined in the various phases: inflationary, radiation, and matter dominated [12,14]. The key idea is that for long-wavelength modes the transitions between the phases are sudden and the Universe will remain in the quantum state it occupied before the transition (valid for all but the highest-frequency modes). However, the creation and annihilation operators that describe the particles in the state are related by a Bogoliubov transformation, so the quantum expectation value of any string of fields is changed. A calculation of the quantum n-point functions suffices to find the spectrum of classical gravitational waves today since the statistical average of the ensemble of classical waves can be related to the corresponding quantum average.

A stochastic spectrum of classical gravitational waves (in terms of comoving wave number $\mathbf{k}$) in the expanding Universe has the form

$$h_\lambda(\tau;\mathbf{k})=A(k)a_\lambda(\mathbf{k})\left[\frac{3j_1(k\tau)}{k\tau}\right], \quad \lambda=+,\times, \tag{5}$$

where $[\cdots]$ is a real solution of the Klein-Gordon equation in a matter dominated universe and $a_\lambda(\mathbf{k})$ is a random variable with statistical expectation value

$$\langle a_\lambda(\mathbf{k})a_{\lambda'}(\mathbf{q})\rangle=k^{-3}\delta^{(3)}(\mathbf{k}-\mathbf{q})\delta_{\lambda\lambda'}. \tag{6}$$

Waves which are still well outside the horizon at the time of matter-radiation equality ($k\tau_1\ll 2\pi$) will give the largest contribution to the CMBR anisotropy today. Calculating Bogoliubov coefficients by matching the fields and first derivatives at τ_2,τ_1 in the limit $k\tau\ll 2\pi$ one derives the prediction for the spectrum of long-wavelength gravitational waves generated by inflation [4,13]

$$A^2(k)=\frac{H^2}{\pi^2 m_{\rm Pl}^2}=\frac{8}{3\pi}v. \tag{7}$$

To make contact with observations one must consider the effect on the CMBR. If one expands the CMBR temperature anisotropy in spherical harmonics

$$\frac{\delta T}{T}(\theta,\phi)=\sum_{lm}a_{lm}Y_{lm}(\theta,\phi), \tag{8}$$

one can present the prediction of a given spectrum of gravitational waves in terms of the a_{lm}. The temperature fluctuation due to a gravitational wave $h_{\mu\nu}$ can be found using the Sachs-Wolfe formula [16].

It is standard to project out a multipole and calculate the symmetric quantity

$$\langle a_l^2\rangle\equiv\left\langle\sum_m |a_{lm}|^2\right\rangle. \tag{9}$$

After some algebra (i.e., see [13]), one finds for waves entering the horizon during the matter dominated era (the results are insensitive to this restriction since the k integral is dominated by waves with $k\approx 2\pi/\tau_0$)

$$\langle a_l^2\rangle=36\pi^2(2l+1)\frac{(l+2)!}{(l-2)!}\int_0^{2\pi/\tau_1}k\,dk\,A^2(k)|F_l(k)|^2, \tag{10}$$

where the function $F_l(k)$ is defined as $[\tau(r)=\tau_0-r]$

$$F_l(k)\equiv\int_0^{\tau_0-\tau_1}dr\left(\frac{d}{d(k\tau)}\frac{j_1(k\tau)}{k\tau}\right)\left[\frac{j_{l-2}(kr)}{(2l-1)(2l+1)}+\frac{2j_l(kr)}{(2l-1)(2l+3)}+\frac{j_{l+2}(kr)}{(2l+1)(2l+3)}\right]. \tag{11}$$

Accounting for the factor of 2 difference between definitions of $A^2(k)$ this agrees with the result of [10], and differs by ≈ 2 with the earlier result of [8].

The calculation of the expectation value $\langle a_l^2\rangle$ is not the end of the story. One must also consider the statistical properties of a_l^2 [8,13,17,18]. Since the a_{lm} are independent Gaussian random variables the probability distribution for each a_l^2, with mean $\langle a_l^2\rangle$, is of a χ^2 form. The confidence levels for a_l^2 are given in terms of the incomplete gamma function. We find for the quadrupole, $\langle a_2^2\rangle=7.74v$ and $a_2^2/\langle a_2^2\rangle=0.63$, 0.32, and 0.23 at the 68%, 90%, and 95% (lower) confidence levels, respectively.

The COBE observations can be summarized for our purposes as a value for the rms quadrupole moment. If one fits to a flat spectrum the quoted value is [1]

$$Q_{\text{rms-PS}}\equiv\left(\frac{a_2^2}{4\pi}\right)^{1/2}=\frac{16.7\pm 4\,\mu\mathrm{K}}{2.73\,\mathrm{K}} \Rightarrow a_2^2=(4.7\pm 2)\times 10^{-10}. \tag{12}$$

Note that the quoted error on $Q_{\text{rms-PS}}$ is Gaussian, while the distribution of a_2^2 is χ^2. This implies that in proceeding from the inferred value of a_2^2 to a value of v we must be careful to properly take into account the resultant statistics which will be far from Gaussian. In particular, the mode of the distribution will be lower than the mean (as is noted [1]). From (12) the quadrupole moment is consistent with gravitational waves resulting from a mean value of $v=6.1\times10^{-11}$. To determine the uncertainty on v we have performed a simple Monte Carlo analysis to find the distribution for v (see Fig. 1). Based on this we can determine both upper and lower limits on the value of v consistent with the observations and the most probable value of v. We find

$$\begin{aligned} &3.7\times10^{-10}>v>2.5\times10^{-12},\ \ 95\%\ \text{C.L.}, \\ &1.5\times10^{-10}>v>2.3\times10^{-11},\ \ 68\%\ \text{C.L.}, \end{aligned} \tag{13}$$

with a maximum likelihood value of $v\approx4\times10^{-11}$.

These limits as quoted require some interpretation. First the 95% upper limit $v<3.7\times10^{-10}$ provides a strict upper limit on the scale of inflation $\approx v^{1/4}M_{\rm Pl}=5.2\times10^{16}$ GeV assuming that the contribution to the quadrupole moment from scalar density perturbations is insignificant. One could scale these results linearly to obtain values for v if the gravitational wave contribution to a_2^2 is not 100%, with some net contribution, or even perhaps a partial shielding by a comparable quadrupole moment from scalar fluctuations, for example.

In this regard it is worthwhile considering what magnitude of quadrupole moment is expected from scalar density perturbations from inflation. By requiring that the induced dipole due to long-wavelength modes not greatly exceed the observed dipole anisotropy one can put an upper limit on all higher multipoles (assuming a flat spectrum). At the 90% confidence level an upper limit of $a_2^2\approx2\times10^{-10}$ has been derived [11]. Equivalently, fitting observed clustering to a primordial fluctuations spectrum [18] one can predict a value of a_2^2, with best fit values in the range $a_2^2\approx(1.9\text{–}9.9)\times10^{-11}$. While these

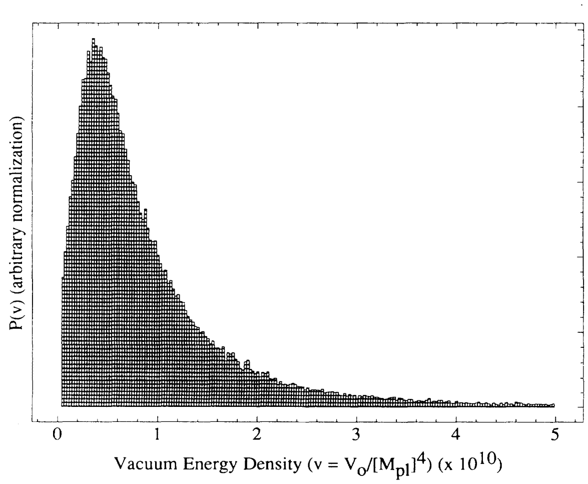

FIG. 1. The distribution for the scale of inflation $v=V_0/M_{\rm Pl}^4$ as determined by Monte Carlo simulation, using the COBE measurements, and assuming the observed quadrupole anisotropy is due to gravitational waves.

estimates are probably consistent with the COBE observation, they also suggest that a major fraction of the observed anisotropy may be due to gravitational waves. Comparison of COBE results with anisotropy measurements on small scales will also be useful.

To what scale of inflation, M, in GeV, do the above limits correspond? From (13) we find, at the 95% confidence level, $1.5\times10^{16}<M<5.2\times10^{16}$ GeV, with the best fit value 2.9×10^{16} GeV. On the other hand, using data from precision electroweak measurements at LEP on the strong- and weak-coupling constants one finds, for minimal SU(5) supersymmetry (SUSY) models with SUSY breaking between M_Z and 1 TeV, that coupling constant unification can occur at a GUT scale M_X in the range $M_X\approx(1\text{–}3.6)\times10^{16}$ GeV [19] or $\sim10^{16}$ GeV in SO(10) models [20]. Unfortunately there are no explicit compelling GUT inflationary scenarios with which one can compare, but generically, unless there is fine tuning, or hierarchies, in a GUT scenario $V_0\approx\kappa M^4$ where $\kappa\approx0.01\text{–}1$ [for example, in a Coleman-Weinberg SU(5) model $\kappa=9/32\pi^2$]. Thus the energy scale of inflation consistent with the observed quadrupole anisotropy coming from gravitational waves can coincide with the estimated GUT scale. We find this possibility both plausible and exciting. At the very least it seems exciting that COBE is sensitive to gravitational waves from inflation at interesting scales (including those which may be related to chaotic inflation [21]).

Since both density perturbations and gravitational wave anisotropies resulting from inflation result in a flat spectrum, with a great similarity in all CMBR multipoles up to at least $l=9$, it will be difficult from CMBR measurements alone to verify whether or not the observed signal is due gravitational waves. How might one hope then to distinguish between these possibilities? The simplest way would be to probe for evidence of a flat spectrum of gravitational waves at smaller wave numbers. At present, the most sensitive gravitational wave detector at shorter wavelengths (periods of $\sim$ years) is also astrophysical in origin, and is based on timing measurements of millisecond pulsars [22–24]. On still smaller wavelengths terrestrial probes, such as the proposed LIGO gravity wave detector [25], are envisaged.

The sensitivity of all such detectors is based on the mean energy density per logarithmic frequency interval in gravitational waves. For waves which come inside the horizon during the matter dominated (MD) era we can utilize (5) and (6). Averaging over many wavelengths/periods, summing over helicities, and also taking the stochastic average, we find

$$k\frac{d\rho_g}{dk}=\frac{k_{\rm phys}^2}{2G}\overline{A^2(k)\left[\frac{3j_1(k\tau)}{k\tau}\right]^2}, \tag{14}$$

where $k_{\rm phys}=k/R(\tau)$. The time evolution factor $[3j_1(k\tau)/k\tau]^2$ is crucial, and in fact implies that the energy density in gravitational waves also redshifts considerably as it comes inside the horizon. Thus the energy density at horizon crossing is smaller than the asymptotic value, a fact which has not been stressed before to our knowledge. Dividing by the critical density today we find

$$(\Omega_g)_{\rm HC} \approx \begin{cases} 16v/9 = 2/3\pi(H_{\rm infl}/M_{\rm Pl})^2, & \text{RD}, \\ v/\pi^2 = 3/8\pi^2(H_{\rm infl}/M_{\rm Pl})^2, & \text{MD}. \end{cases} \qquad (15)$$

The result at horizon crossing in a radiation dominated (RD) epoch results from the factor of $3j_1(k\tau)/k\tau$ above being changed to $j_0(k\tau)$. Waves which come inside the horizon during the radiation dominated era will redshift with one extra power of R compared to matter during the matter dominated era. Thus their contribution to Ω today will be suppressed compared to their contribution at horizon crossing by the factor $\rho_{\rm rad}/\rho_c = 4\times10^{-5}h^{-2}$, where the Hubble constant today is $100h$ km/sec Mpc. As a result, we find that such waves today, taking $v < 3.7\times10^{-10}$, form a stochastic background with $\Omega_g < 2.6\times10^{-14}h^{-2} < 10^{-13}$ ($h > 0.5$).

The waves for which the millisecond pulsar timing and future interferometer measurements are sensitive entered the horizon during the radiation dominated era. The present limit, at the 68% confidence level, from pulsar timing data is $\Omega_g < 9\times10^{-8}$ [24]. This can improve in principle as the measuring time to the fourth power [22,23] but, even in the most optimistic case, observations over a period of perhaps a century would be required to uncover such a signal. The expected energy density is also about 2 orders of magnitude below the optimum projected capabilities of future terrestrial detectors.

Thus, prospects look grim in the short term for detecting such a gravitational wave background directly elsewhere. Barring a very refined measurement of high multipoles in the CMBR anisotropy we may have to await confirmation at accelerators, proton decay detectors, or inferences from other CMBR and large-scale structure measurements before we can say whether COBE has discovered the first evidence for GUTs, supersymmetry, or at the very least, gravitational waves.

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