$\frac{4}{5}$ Kolmogorov Law for Statistically Stationary Turbulence: Application to High-Rayleigh-Number Bénard Convection

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(Received 10 April 1992)

The Kolmogorov relation for the third-order moments of the velocity differences is generalized for the case of statistically steady turbulence and applied to the Bénard convection problem. The predicted temperature and velocity spectra are $E_T \approx k^{-7/5}$ and $E \approx k^{-11/5}$, respectively. At the smaller scales, in the dissipation range of the temperature fluctuations, the Kolmogorov range where most of the energy is dissipated is predicted. The new set of scaling exponents, which can be observed in the experiments in the small-aspect-ratio convection cells, is derived.

PACS numbers: 47.25.Cg

In 1941, Kolmogorov derived his celebrated relation for the third-order structure functions in decaying homogeneous and isotropic turbulence [1]:

$$S_3 = \langle [u(X)-u(X+x)]^3\rangle = -\frac{4}{5}\epsilon x + 6\nu\frac{dS_2}{dx}, \quad (1)$$

where $u(X)$ is the x component of the velocity field $\mathbf{v}$, x is the displacement in the x direction, and $\epsilon = \nu\langle(\partial v_i/\partial x_j)^2\rangle \sim 1$. The correlation function is $S_2 = \langle [u(X)-u(X+x)]^2\rangle$. The relation (1) is the consequence of the Navier-Stokes equations for an incompressible fluid, and the dissipation rate ϵ in the Kolmogorov derivation is defined as $\langle \partial v^2/\partial t\rangle = -2\epsilon$. In a statistically steady flow driven by the force $\mathbf{f}$, $\langle dv^2/dt\rangle = 0 = -2\epsilon + 2\langle \mathbf{f}\cdot\mathbf{v}\rangle$ and, in general, the Kolmogorov relation (1) must be modified. The Navier-Stokes equations driven by the force $\mathbf{f}$ are

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 v_i}{\partial x_j^2} + f_i, \quad (2)$$

with $\nabla\cdot\mathbf{v}=0$. For simplicity we consider a statistically isotropic force so that $\langle f_\alpha v_\alpha\rangle = \langle f_\beta v_\beta\rangle$ for $\alpha\neq\beta$. In this paper the summation over the Greek indices is not assumed. Applying the Kolmogorov derivation, the details of which are given in Refs. [1-3], to Eq. (2), the following relation is readily obtained:

$$\frac{1}{6x^4}\frac{\partial x^4 S_3}{\partial x} + \langle u(X)f(X+x)\rangle + \langle u(X+x)f(X)\rangle = \frac{\nu}{x^4}\frac{\partial}{\partial x}x^4\frac{\partial S_2}{\partial x}, \quad (3)$$

where f is the x component of the force. Integrating (3) leads to

$$S_3 = -\frac{6}{x^4}\int_0^x y^4[\langle u(X)f(X+y)\rangle + \langle u(X+y)f(X)\rangle]dy + 6\nu\frac{\partial S_2}{\partial x}. \quad (4)$$

Taking into account that $\langle u(x)f(x)\rangle = \frac{1}{3}\epsilon$ expression (4) can be recast in a more familiar form:

$$S_3 = -\frac{4}{5}\epsilon x + \frac{6}{x^4}\int_0^x y^4\langle \Delta u\Delta f\rangle dy + 6\nu\frac{\partial S_2}{\partial x}, \quad (5)$$

where $\Delta f = f(X) - f(X+x)$. This relation is exact. It is clear that if the energy source acts at the largest scales only, so that the Fourier transform $f(k)=0$ for $k>k_0 \to 0$, then relations (5) and (1) are identical for small enough values of the displacement x. However, if the field $\mathbf{f}$ is correlated with $\mathbf{v}$ at all scales, then relation (1) is grossly incorrect.

In what follows the generalized Kolmogorov relation (5) will be applied to the problem of Bénard convection between infinite plates separated by a distance L. The mean temperature difference between top and bottom plates is $\Delta\theta = 2\Delta$. The equations of motion for the velocity field $\mathbf{v}$ and for the temperature fluctuations T from the mean temperature profile $\theta(z)$ in the convection cell are

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 v_i}{\partial x_i^2} + \alpha g T\delta_{i3}, \quad (6)$$

$$\frac{\partial T}{\partial t} + v_j\frac{\partial T}{\partial x_j} = \kappa\frac{\partial^2 T}{\partial x^2} + \kappa\frac{\partial^2\theta}{\partial x_i^2} - v_3\frac{\partial\theta}{\partial x_3}. \quad (7)$$

Here α is the thermal expansion coefficient and g is the gravitational acceleration. The mean temperature gradients in the parts of the cell outside the close-to-the-wall thermal boundary layer are very small. This allows us to assume that $T(x,t)$ is a statistically isotropic and homogeneous process. In this part of the cell the scalar dissipation rate $2N = 2\kappa\langle(\partial T/\partial x_i)^2\rangle = -2\langle v_3 T\rangle\,\partial\theta/\partial z$. Again, applying the procedure developed for the problem of a decaying passive scalar [2] to Eq. (7) we readily derive

$$\frac{1}{2x^2}\frac{\partial x^2 S_3^T}{\partial x} - [\langle v_3(X)T(X+x)\rangle + \langle v_3(X+x)T(X)\rangle]\frac{\partial\theta}{\partial z} - \frac{\kappa}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial S_2^T}{\partial x} = 0,$$

where $S_3^T = \langle\Delta u(\Delta T)^2\rangle$, $S_2^T = \langle(\Delta T)^2\rangle$, and $\Delta T = T(X) - T(X+x)$. This relation can be rewritten as

$$S_3^T = -\frac{4}{3}Nx + \frac{2}{x^2}\int_0^x y^2\langle\Delta v_3\Delta T\rangle dy\frac{\partial\theta}{\partial z} + 2\kappa\frac{\partial S_2^T}{\partial x}. \quad (8)$$

Expression (8) is an exact consequence of Eq. (7) in the part of the cell outside the thermal boundary layer, adjacent to the top and bottom plates. We assume that there the small-scale turbulence is isotropic and homogeneous. On first glance this assumption cannot be plausible since, as seen from Eq. (6), the energy is pumped into the z component of the velocity field only. However, pressure tends to rapidly redistribute the energy between different components of the velocity field leading to the isotropization of the small-scale turbulence. It follows from Eq. (6) and the incompressibility condition that the pressure term can be split into two: $p=p_c+p_1$, where $p_c=-(\nabla_i\nabla_j/\nabla^2)v_iv_j$ as in free turbulence and $p_1=\alpha g(\nabla_3/\nabla^2)T$ so that

$$-\left\langle v_\beta\frac{\partial p_1}{\partial x_\beta}\right\rangle=-\alpha g\left\langle v_\beta\frac{\nabla_3\nabla_\beta}{\nabla^2}T\right\rangle=O(\alpha g\langle v_\beta T\rangle)$$

is an effective energy source for the components v_β with $\beta\neq 3$. Based on these considerations we write an approximate equation valid in the central region of the cell:

$$\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p_c}{\partial x_i}+\nu\frac{\partial^2 v_i}{\partial x^2}+\frac{1}{3}\alpha g T n_i\quad(n_i=1)\,. \tag{9}$$

Equation (9) is convenient since it enables us to use the generalized Kolmogorov relation (5). However, the conclusions of the scaling theory developed below do not change if the exact expressions for the sources coming from (6) are used. Thus, we have from (2), (6), and (9)

$$\langle(\Delta u)^3\rangle=-\frac{4}{5}\epsilon x+\frac{2\alpha g}{x^4}\int_0^x y^4\langle\Delta u\Delta T\rangle dy+6\nu\frac{\partial S_2}{\partial x}\,. \tag{10}$$

Relations (8) and (10) will be used to develop a scaling theory of convection in the limit $\nu\approx\kappa\approx 0$. It will be shown below that the first term on the right-hand side of (10) is small at large enough scales. Thus, neglecting it for the time being and taking into account that expression (8) is dominated by the $O(Nx)$ contributions, the following scaling relations are readily derived: $\langle(\Delta T)^2\rangle\approx N^{4/5}\times g^{-2/5}x^{2/5}$, $\langle(\Delta u)^2\rangle\approx g^{4/5}N^{2/5}x^{6/5}$. The corresponding spectra are

$$E_T\approx N^{4/5}g^{-2/5}k^{-7/5},\quad E\approx N^{2/5}g^{4/5}k^{-11/5}\,. \tag{11}$$

In the high-Ra flow the heat is mainly transferred by the velocity fluctuations so that the effective scale-dependent transport coefficients $\nu(k_i)\approx\kappa(k_i)$ and the turbulent viscosity is estimated as [4]

$$\nu(k)\approx[E(k)/k]^{1/2}\approx N^{1/5}g^{2/5}k^{-8/5}\,. \tag{12}$$

This leads to an important conclusion: The scalar dissipation rate $N\approx\kappa\int_0^{k_d}E_T(k)k^2dk=\kappa(k)\int_0^k E_T(k)k^2dk$ $=$const is scale independent. This means that in thermal convection the flux of $\langle T^2\rangle$ but not that of energy is constant in the wave-number space. Substituting the above estimates into (7) the expression for the dissipation scale of the temperature fluctuations is readily obtained:

$$l_d\approx\nu^{5/8}g^{-1/4}N^{-1/8}\,. \tag{13}$$

From the definition of the effective diffusivity, we have

$$N=-\langle vT\rangle\frac{\partial\theta}{\partial z}\approx\kappa(k_i)\left(\frac{\partial\theta}{\partial z}\right)^2\approx\frac{H^2}{\kappa(k_i)}\,, \tag{14}$$

where $k_i\approx 1/L$, $H=$const is the heat flux through the cell, and $\kappa(k_i)$ is the turbulent diffusivity defined by (12) and (14). Now we introduce dimensionless variables, setting $\Delta=L=\nu=\kappa=\alpha=1$ so that $g=\mathrm{Ra}$ and $H=N_u$ where the dimensionless heat flux N_u is called the Nusselt number. It follows from (12) and (14) that

$$N\approx N_u^{5/3}\mathrm{Ra}^{-1/3}\,. \tag{15}$$

The heat flux is estimated readily through the width of the thermal boundary layer l_B where the mean temperature θ varies from zero to $\theta\approx\Delta$, its value at the center of the cell: $H\approx\kappa\Delta/l_B=$const. In the dimensionless units this gives $N_u\approx l_B^{-1}$. In the simplest situation $l_B\approx l_d$, which is the smallest scale of the temperature fluctuations in turbulence, and using (13) the following relation is derived:

$$N_u\approx\mathrm{Ra}^{5/19}\,. \tag{16}$$

The root mean square temperature and velocity found from the corresponding spectra (11) are

$$T_{\mathrm{rms}}\approx\mathrm{Ra}^{-3/19},\quad v_{\mathrm{rms}}\approx\mathrm{Ra}^{8/19}\,. \tag{17}$$

Expressions (16) and (17) are valid only when $\epsilon x\ll\alpha g\langle\Delta T\Delta u\rangle x$. In the dimensionless variables $N\approx\mathrm{Ra}^{2/19}$, $l_d\approx\mathrm{Ra}^{-5/19}$, $\epsilon\approx\mathrm{Ra}^{20/19}$ and this inequality reads $\mathrm{Ra}^{4/19}x^{4/5}\gg 1$ which holds for $x\gg l_d$. Thus, in the range of scales $l_d<l<L$ the scaling relations (16),(17), and the "nonclassical" exponents (11) are observed. It follows from (8) that the scales $l<l_d$ correspond to the temperature dissipation range where the amplitude of the temperature fluctuations is negligibly small. However, we will show below that this is the range where most of the energy is transferred to and then dissipated into heat. The rate of the energy production P, derived from the equations of motion (6), is $P=\alpha g\langle v_3T\rangle\approx N_u\mathrm{Ra}\approx\mathrm{Ra}^{24/19}$. If we calculate the dissipation rate ϵ using the spectrum (11) in the interval $l_d<l<L$ we find easily that $\epsilon\approx\mathrm{Ra}^{21/19}$ and, thus, the energy balance evaluated in this interval where relations (11), (13), (16), and (17) hold is not satisfied since more energy is produced than dissipated. In the temperature dissipation range where $l<l_d$ we neglect the temperature fluctuations and find from relation (1) or (10) that the energy balance is in fact not violated since there exists another inertial range of scales $l_K<l<l_d$, characterized by the Kolmogorov spectrum: $E(k)\approx\epsilon^{2/3}k^{-5/3}$ with $\epsilon=O(\mathrm{Ra}^{24/19})$ which is equal to the rate of the turbulence production. In this case $l_K\approx\mathrm{Ra}^{-6/19}$. It is important to emphasize that the

velocity fluctuations from this interval do not contribute to the heat transfer process. In addition, this range is extremely narrow since $l_d/l_K = O(\mathrm{Ra}^{1/19})$ and the experimental observation of the Kolmogorov spectrum of the velocity fluctuations might be very difficult. Thus, the scaling relations (16) and (17) and the spectra predicted in this work should always be observed in convection experiments provided the estimate for the width of the thermal boundary layer $l_B \approx l_d$ given by (15) is not modified by some more complicated dynamical phenomena. For example, if l_B is found from the relation $\alpha g T_{\mathrm{rms}} l_B^3/\kappa\nu \sim 1$ then formulas (11) and (14) give

$$N_u \approx \mathrm{Ra}^{2/7}, \quad T_{\mathrm{rms}} \approx \mathrm{Ra}^{-1/7}, \quad v_{\mathrm{rms}} \approx \mathrm{Ra}^{3/7}. \tag{18}$$

Since $l_d \approx \mathrm{Ra}^{-2/7}$ the energy production and dissipation rates scale as $P \approx \mathrm{Ra}^{9/7}$ and $\epsilon \approx \mathrm{Ra}^{38/35}$, respectively. In this case, too, more energy is produced than dissipated in the 1.4 range given by (11). Thus, a very narrow Kolmogorov range, where the balance of the energy is dissipated, is to be expected in the interval $l_K < l < l_d$. There $\epsilon = P$ and $l_K \approx \mathrm{Ra}^{-9/28}$.

The ratio $l_d/l_K \approx \mathrm{Ra}^{1/28}$, which makes the experimental identification of the Kolmogorov range equally difficult. Relations close to (18) have been observed in experiments on high-Ra-number convection in large-aspect-ratio ($A > 0.5$) cells where the persistent large-scale vortex ("wind") strongly influences stability and the length scale of the thermal boundary layer [5–7]. However, it has been noticed [8] that in small-aspect-ratio cells this large-scale motion becomes unstable and one can assume that when $A \ll 1$ the destabilizing influence of the wind is negligibly small due to shear layer instability. In this case crossover to the scaling behavior (16) and (17) can be expected. Indeed, it has been reported by Threlfall [9] that in a cell with $A = 0.14$ the Nusselt number $N_u \approx \mathrm{Ra}^{0.265}$, which is extremely close to prediction of this work (16) ($\frac{5}{19} \approx 0.263$). The experimental evidence is not conclusive and much more extensive experimental and numerical work is needed to verify all other predictions derived in this paper to come to more definite conclusions about realizability of relations (11), (16), and (17) in real-life flows.

Another interesting outcome of this work is related to the single-point probability distributions of temperature fluctuations in thermal convection. It has been reported [5,7] that the rise of the nonclassical exponents (18) is accompanied by dramatic changes in the shape of probability distribution functions (PDFs) of temperature fluctuations: In the regime in which scaling relations (18) are observed the PDFs have exponential shapes. This behavior of the PDFs has been explained in terms of the changes in the turbulence production mechanisms [10]: The strong wind induces the instability of the boundary layer leading to formation of very energetic plumes or thermals emitted from the boundary layer with characteristic velocity, close to the velocity of the wind. In the absence of characteristic velocity, as in the low-Ra-number convection, the PDFs must be close to Gaussian [10]. If the theory developed here is applied to convection in small-aspect-ratio cells, then the predicted PDFs of the single-point temperature fluctuations must be close to the Gaussian since no characteristic velocity scale is assumed in the derivation of (11), (16), and (17).

To conclude this paper I would like to mention that the scaling exponents (16) and (17) derived in this work are the direct consequence of the nonclassical spectra (11). If the dynamics are dominated by the turbulence dissipation rate ϵ then the "classical" $\frac{1}{3}$ relation is readily recovered from the dynamical picture considered in this work. Indeed, in this case $\epsilon \approx \alpha g \kappa (k_i) \partial\theta/\partial z \approx \mathrm{Ra} N_u$. In the Kolmogorov turbulence the dissipation scale $l_d \approx \epsilon^{-1/4} \approx (\mathrm{Ra} N_u)^{-1/4}$. Assuming as before that $l_d \approx l_B$ and taking into account that $N_u \approx l_B^{-1}$, the "classical" expression $N_u \approx \mathrm{Ra}^{1/3}$ is readily obtained. However, according to the above estimates the Kolmogorov spectrum cannot be observed over the wide range of scales in the high-Rayleigh-number convection and, thus, the range of applicability of the $\frac{1}{3}$ law is rather narrow.

It is a pleasure to acknowledge stimulating discussions with A. Libchaber, M. Nelkin, L. Sirovich, L. Kadanoff, B. Shraiman, and L. Smith.

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