Determination of the Strong Coupling Constant from the Charmonium Spectrum

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Lattice gauge theory techniques have recently achieved sufficient accuracy to permit a determination of the strong coupling constant from the $1P$-$1S$ splitting in the charmonium system, with all systematic errors estimated quantitatively. The present result is $\alpha_{\overline{\rm MS}}(5\ {\rm GeV})=0.174\pm0.012$, or, equivalently, $\Lambda^{(4)}_{\overline{\rm MS}}=160^{+47}_{-37}$ MeV ($\overline{\rm MS}$ denotes the modified minimal subtraction scheme).

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A central task in understanding quantum chromodynamics (QCD) is the determination of its coupling constant, g^2. The Particle Data Group quotes values for $\alpha_s(5\ {\rm GeV})\equiv g^2/4\pi$ in the range 0.18–0.22 [1]. Recent measurements at the CERN e^+e^- collider LEP yield values in the range 0.20–0.24 [2]. Most perturbative determinations of g^2 contain nonperturbative contaminations which become small only at high energies. On the other hand, high-energy determinations yield g^2 at lower energies only imprecisely. Lattice gauge theory calculations provide a nonperturbative means of determining the strong coupling constant from low-energy quantities.

In principle, any lattice calculation of a mass or energy E allows a determination of the strong coupling constant. The lattice calculation yields the dimensionless quantity aE, where a is the lattice spacing which is determined by comparing aE with the experimentally measured value for E. The bare lattice coupling constant g_0^2 at scale a may then be converted into one of the more familiar definitions of the coupling constant using known perturbative results [3,4]. In practice, most existing lattice calculations contain systematic errors which are difficult to analyze quantitatively. Consider, for example, the obvious case of the proton mass. Lattice calculations have not yet been done with quark masses as light as their physical values. Chiral perturbation theory calculations [5] indicate that at pion masses of around 400 MeV, where lattice calculations are often done, the proton mass is reduced by a term (of order m_π^3) of around 100% of its physical, light pion value. Similarly, the most accurate lattice calculations to date have been done ignoring the effects of sea quarks. Some chiral quark model calculations [6] estimate that the proton mass may be altered as much as 30% by the effects of the strange quarks in the sea, let alone the light quarks. Whether or not these calculations are quantitatively reliable, the point is that the approximation of ignoring the sea quarks (the "quenched" approximation) introduces potentially large systematic errors for the light hadrons which are difficult to analyze and control.

Heavy quark systems offer the best opportunity for determining the strong coupling constant with present day lattice calculations [7]. For these systems no extrapolation to light valence quark masses is necessary, and errors arising from the omission of sea quarks and also from the finiteness of the lattice spacing may be systematically analyzed and quantitatively estimated with some input from phenomenology, as we discuss below. As lattice calculations improve, the phenomenological aspects of the corrections and error analysis will ultimately be removed. A rigorous program to extract the strong coupling constant from lattice gauge theory entirely from first principles is being formulated by Lüscher *et al.* [8].

The cleanest quantity in heavy-quark systems from which to extract the strong coupling constant is the splitting between the spin averaged masses of the $1S$ and $1P$ states. This splitting is insensitive to errors in spin-dependent interactions which are induced by the finite size of the lattice spacing. It is also known to be quite insensitive to any errors in the definition of the quark mass, since the $1P$-$1S$ splittings in the ψ and Υ systems are almost identical. Higher-order finite-lattice-spacing errors such as those resulting in effective p^4 interactions and four-fermion interactions may be easily analyzed perturbatively using potential models and Coulomb gauge lattice wave functions, and if necessary removed by corrections to the lattice action.

We have calculated this splitting using standard Monte Carlo techniques at three lattice spacings (or, equivalently, three values of $\beta\equiv6/g_0^2$). The smallest lattice spacing used corresponds to $\beta=6.1$ on lattice volumes of 24^4 with 25 gauge configurations separated by 8000 pseudo-heat-bath sweeps. The larger lattice spacings correspond to $\beta=5.9$ on volumes of 16^4 and $\beta=5.7$ on volumes of $12^3\times24$, each with 25 configurations separated by 2000 pseudo-heat-bath sweeps.

The Wilson action for fermions was used with the $O(a)$ correction term, $-(i/2)c\bar\psi\Sigma_{\mu\nu}F_{\mu\nu}\psi$, added [9]. [Addition of this operator to the action suffices to correct $O(a)$ errors for Wilson fermions.] The coefficient c was set to 1.4 rather than its tree-level value $c=1$ on the basis of mean-field theory estimates of the higher-order corrections. For nonrelativistic fermions, it contributes mainly an additional $\sigma\cdot B$ interaction to the quarks. The spin averaged level splitting on which this paper is based is expected to be very insensitive to this correction. On the other hand, such spin splittings as the ψ-η_c splitting which we have also investigated are very sensitive to it

[10].

The lattice spacing for each value of β was obtained by calculating in lattice units the difference of the spin averaged mass of the $1S$ states (the J/ψ and the η_c) and the mass of the recently discovered spin singlet $1P$ state (the h_c) [11], and then comparing with the experimentally measured splitting, $M_{h_c}-(3M_\psi+M_{\eta_c})/4=458.6\pm0.4$ MeV. Coulomb gauge wave functions were used to create and destroy the meson states to reduce errors from excited states. Since quarks in the charmonium system are nonrelativistic, it is not surprising that these waves give a very good approximation to the full states. No indications of contamination of the two point functions from excited states were seen after a separation in Euclidean time of one lattice spacing for the S states and two lattice spacings for the P states. The analysis was based on separations of at least three lattice spacings. For more details, see Ref. [10].

From the lattice spacing a and the bare lattice coupling constant g_0^2, the $\overline{\rm MS}$ (modified minimal subtraction) coupling at scale π/a may be obtained using the one-loop perturbative formula $g_{\overline{\rm MS}}^{-2}(\pi/a)=g_0^{-2}(1-\frac{1}{3}g^2)+0.025$ [3,4]. The background field calculation of this correction [4] shows that it is dominated by the term in parentheses, which is the perturbative expectation value of the plaquette $\langle {\rm Tr}U_P\rangle$. We have therefore substituted the known nonperturbative value of the plaquette at each β to correct for higher-order effects in the relation between the bare and $\overline{\rm MS}$ coupling constants [12], using

$$\frac{1}{g_{\overline{\rm MS}}^2(\pi/a)}=\frac{1}{g_0^2}\langle {\rm Tr}U_P\rangle_{\rm MC}+0.025\,. \qquad (1)$$

(The values for $\langle {\rm Tr}U_P\rangle_{\rm MC}$ are 0.549, 0.582, and 0.605 at $\beta=5.7$, 5.9, and 6.1, respectively.) This yields an additional correction to the $\overline{\rm MS}$ coupling constant (11% of the final value) which is much smaller than the one-loop correction (about 44%) but not negligible.

Figure 1 shows the Coulomb gauge wave function of the ψ meson calculated on the 24^4, $\beta=6.1$ lattices. It is probably not controversial that finite volume errors are negligible for the lattice sizes used. To check this, a low statistics calculation was done at $\beta=6.1$ on a volume of 16^4 which yielded a value of the $1P$-$1S$ splitting (10–20)% larger than that on the 24^4 lattices. Figure 1 shows that the wave function of the ψ is about a factor of 5 smaller at a distance of twelve lattice spacings (halfway across the 24^4 lattice) than it is at eight lattice spacings (halfway across the 16^4 lattice). Assuming that finite volume errors fall roughly as the square of the wave function halfway across the lattice leads to the conclusion that the errors on the 24^4 lattice are under a percent in the spin splitting and therefore in $\Lambda_{\overline{\rm MS}}$. This implies errors in α of a fraction of a percent.

We have already noted that the spin averaged $1P$-$1S$ splitting is very insensitive to the $O(a)$ errors of the Wilson quark action. Further, we have used a corrected action which minimizes these errors. It is therefore to be expected that the most important finite-lattice-spacing errors remaining will be of order a^2. To test for the size of these, the calculation was performed at three lattice spacings. From the $1P$-$1S$ splitting, we obtained $a^{-1}=1.15(8)$, 1.78(9), and 2.43(15) GeV at $\beta=5.7$, 5.9, and 6.1, respectively. The errors in parentheses are statistical only. Using Eq. (1) and the parametrization for α_s of the Particle Data Group, values for $\Lambda_{\overline{\rm MS}}^{(0)}$ were obtained for each lattice spacing. In Fig. 2 the results are plotted as a function of a^2. The value of $\Lambda_{\overline{\rm MS}}^{(0)}=234$ MeV after extrapolation to $a^2=0$ is about 4% larger than its value at $\beta=6.1$. The statistical errors are not small enough to distinguish between possible functional forms for the finite a errors (e.g., a, a^2, or a^4), and we rely on theoretical prejudice in making the extrapolation in a^2. We therefore take the 4% difference between the extrapolated value and the value at $\beta=6.1$ as a contribution to the sys-

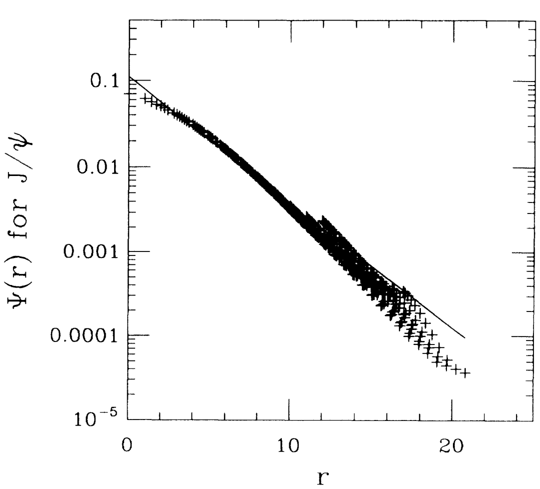

FIG. 1. The wave function of the ψ meson.

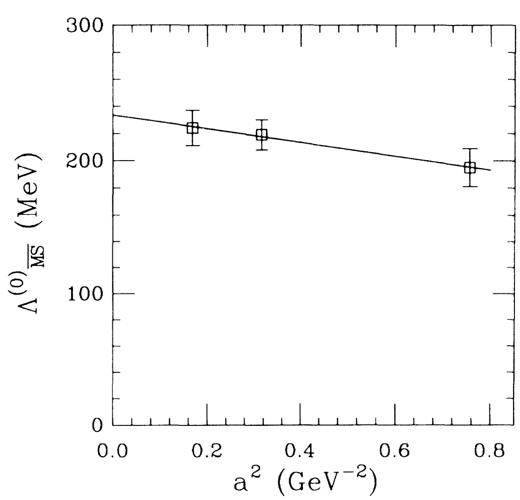

FIG. 2. $\Lambda_{\overline{\rm MS}}^{(0)}$ as a function of a^2.

tematic uncertainty. The next step in removing this small contribution to the uncertainty will be the evaluation of the known discretization errors present in the lattice action with the Coulomb gauge wave functions obtained from the lattice calculations, to verify or improve the functional form of the extrapolation in Fig. 2.

The final and largest source of uncertainty arises from the conversion from the zero-light-quark running coupling constant of the lattice calculation to the four-quark running coupling of the real world. It is more convenient to discuss this uncertainty in terms of g^2 rather than Λ. The fact that the system is nonrelativistic implies that the dominant effects of the omission of sea quarks lie in their effect on the static potential. We use this effect as an estimate of the corrections and uncertainty arising from this source. The adjustment of the bare parameters of the lattice action to reproduce experimental physics is really the adjustment of the parameters in the effective action at the physics momentum scale to be correct. Since the β function of the quenched (zero light quark) lattice theory is slightly too large, the short distance coupling constant of the quenched lattice theory must be slightly too small. It is this discrepancy which must finally be estimated. In perturbation theory, requiring that the effective actions of the lattice and the real world match at the physics scale μ_1 requires approximately that the running coupling constants match at this scale. The required difference in the running couplings at scale μ_2 is then given by

$$\Delta g^{-2}=\int_{\mu_1}^{\mu_2} d\ln q^2[\beta_0^{(n_f)}-\beta_0^{(0)}+(\beta_1^{(n_f)}-\beta_1^{(0)})g^2+\cdots], \tag{2}$$

where

$$\beta_0^{(n_f)}=(11-\tfrac{2}{3}n_f)/16\pi^2,$$

$$\beta_1^{(n_f)}=(102-\tfrac{38}{3}n_f)/(16\pi^2)^2.$$

From potential models we know that the typical momentum transfer in J/ψ mesons is around $\mu_1\approx 400$ MeV. Most of the required correction arises from the large μ region where perturbation theory is valid, and is therefore reliably given by Eq. (2). Integrating the correction from $\mu_2=5$ GeV down to 750 MeV, the scale of typical quark momentum in the J/ψ, and making the appropriate change from the four-quark to the three-quark β function at the charmed quark mass, gives a contribution to the correction of $\Delta g^{-2}=-0.080$. (g_V, the coupling constant defined from the heavy-quark potential, was used in the integration.) This is certainly an underestimate of the correction, since the β function is quite convergent in this region and 750 MeV is the largest momentum scale in the systems we are studying. If we take $\mu_1\approx 400$ MeV as the best guess for the appropriate matching scale, we obtain an additional correction of $\Delta g^{-2}=-0.060$. This large contribution from a small region of integration arises from the fact that 400 MeV is near the fictitious pole in the two-loop perturbative coupling constant, causing g_V^2 to begin to blow up near the end of the integration region. (At $\mu_1\approx 400$ MeV, we obtained $g_V^2=45$.) Although the true contribution of the low-energy region to the coupling constant correction cannot be reliably estimated perturbatively, it is certainly not divergent, so this is almost certainly an overestimate of the true correction. Since the effects of light quarks on the potential are relatively mild both at short distances (known from the perturbative β functions) and at large distances (known from comparing the string tensions obtained from Regge phenomenology and from quenched lattice calculations), it is implausible that their effects in the intermediate region are overwhelmingly severe. We therefore take these two estimates as upper and lower bounds on the true correction, giving $\Delta g^{-2}=-0.110\pm 0.030$. The approximate size of this correction, 24%, clearly makes sense, since if the comparison scale 5 GeV is replaced by an asymptotically large value, μ_2, the obtained correction must approach $(\beta_0^{(n_f)}-\beta_0^{(0)})/\beta_0^{(0)}=\frac{2}{3}(n_f/11)\approx 24\%$ for $n_f=4$. Over the next few years, this largest source of uncertainty will be eliminated by the inclusion of sea quarks in lattice calculations. In the short term, it may be clarified and, we hope, reduced, with the use of potential models and a study of the static potentials of the quenched and unquenched lattice theories.

Our final result is

$$\alpha_{\overline{\rm MS}}(5\ {\rm GeV})=0.174\pm 0.012\,. \tag{3}$$

This corresponds to $\Lambda_{\overline{\rm MS}}^{(4)}=160^{+47}_{-37}$ MeV, using the parametrization of the Particle Data Group. The known corrections and uncertainties are summarized in Table I. Each correction ϵ_i is defined to mean that the corrected coupling α_i is given by $\alpha_i=(1+\epsilon_i)\alpha_{i-1}$. The uncertainty is dominated by the uncertainty in relating the zero-quark coupling constant of the lattice calculation to the four-quark coupling constant of the real world. Over the next few years, Monte Carlo simulations including the effects of sea quarks will eliminate this currently dominant contribution. They will leave residual errors of only

TABLE I. Summary of corrections and uncertainties in the determination of $\alpha_{\overline{\rm MS}}^{(4)}$. Note that all of the corrections have the same sign, raising the obtained value of α.

Source	Correction	Uncertainty
$g_0^2\to g_{\overline{\rm MS}}^2$ (one loop, Refs. [3,4])	44%	...
$g_0^2\to g_{\overline{\rm MS}}^2$ (Monte Carlo correction, Ref. [12])	11%	...
$g^{(0)}\to g^{(4)}$	24%	6.6%
Statistics	...	2%
Finite lattice spacing	1%	1%
Finite volume	...	...

(2–3)%, much smaller than those currently associated with the convergence of perturbation theory.

As lattice and perturbative determinations of the strong coupling constant improve, it will eventually become necessary to replace the $\overline{\text{MS}}$ coupling constant with a standard of comparison defined from some physical process. This will insure that uncertainties such as those associated with the convergence of perturbation theory, which are intrinsic to only one regulator, not be propagated to all determinations. The process used for the standard of comparison should be one which is easy to calculate in all regulators. The heavy-quark potential at short distances is a likely candidate. We have used perturbation theory more than once in the above calculation, but it plays an essential role only in the conversion of the lattice coupling constant to the $\overline{\text{MS}}$ coupling constant. We intend to replace this part of the calculation with a nonperturbative determination of the coupling utilizing the heavy-quark potential in a future publication. We have therefore not attempted the difficult task of estimating the uncertainty arising from the truncation of perturbation theory, and leave that additional piece of error estimation to the reader.

Recently, a similar calculation has been performed for both the ψ and Υ systems using the nonrelativistic formulation of lattice fermions [13]. For the Υ system, the systematic errors and corrections are quite different from the ones reported here. The results are compatible.

Extrapolating to the mass of the Z, we obtain $\alpha_{\overline{\text{MS}}}(M_Z)=0.105\pm0.004$. This is a little more than 2σ below the combined results from LEP: $\alpha(M_Z)=0.119\pm0.006$ [2].

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