Analytic Solution of the Random Ising Model in One Dimension

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An analytic expression is derived for the Lyapunov exponents of the product of random transfer matrices related to the Ising model with quenched disorder in one dimension. We find a deterministic map which transforms the original system into a new one with zero external field and constant coupling. The free energy and the rate of correlation decay are thus obtained in terms of an exponentially convergent series. Our results can be generalized to the product of random matrices with nonzero entries.

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Products of random matrices appear as the natural tool for the study of disordered systems as well as of chaotic dynamical systems [1,2]. In this context, the calculation of the spectrum of the Lyapunov characteristic exponents (LCE) has great relevance. However, exact solutions are known only for very particular cases [3,4], even if some nonperturbative analytic results have been recently obtained (the so-called microcanonical method [5] and the cycle expansion of the Ruelle zeta function [6]). Here we present a method which allows one to determine the LCE in terms of an exponentially convergent series, for positive random matrices. Although we are not able to find a closed expression of the analytic function determined by the series, its terms can be computed up to obtain the LCE spectrum with the desired precision.

The object of our study is the spectrum of the product $\mathbf{G}_N = \prod_{i=1}^{N} \mathbf{T}(i)$ of N independent identically distributed $d \times d$ random matrices $\mathbf{T}(i)$. The Lyapunov exponents $\lambda_1 \geq \lambda_2, \ldots, \lambda_d$ are defined as the logarithm of the absolute values of the eigenvalues of the matrix $(\mathbf{G}_N^\dagger \mathbf{G}_N)^{1/2N}$ in the asymptotic limit $N \to \infty$. The multiplicative ergodic theorem of Oseledec [7] ensures that this limit spectrum exists and is nonrandom for almost all realizations (i.e., all realizations a part of a set of zero probability measure) of the $\mathbf{G}_N$'s, which are themselves stochastic variables. In particular, the maximum LCE is the rate of exponential divergence of the norm of a generic vector $\mathbf{z} \in R^d$ under the successive applications of $\mathbf{T}(i)$:

$$\lambda_1 = \lim_{N \to \infty} \frac{1}{N} \ln \frac{|\mathbf{G}_N \mathbf{z}|}{|\mathbf{z}|} = \lim_{N \to \infty} \frac{1}{N} \ln \overline{\frac{|\mathbf{G}_N \mathbf{z}|}{|\mathbf{z}|}}, \tag{1}$$

where the overbar indicates the average over the different disorder realizations. In the context of statistical mechanics, this property is known as self-averaging.

For the one-dimensional Ising model with random fields h and/or random nearest-neighbor couplings J, one deals with 2×2 matrices, and the computation of the two Lyapunov exponents gives the free energy and the decay rate of the spin-spin correlation. In practice we are able to find a mapping [8] to transform an Ising model into a new one with smaller random fields and couplings. Iterating this mapping, one tends to a fixed point corresponding to a system with zero external field and constant coupling which has a trivial solution. With the same method, one can obtain the LCE spectrum of the product of generic positive 2×2 random matrices since they can always be written as transfer matrices of an appropriate Ising model.

To be explicit, consider an Ising model on a one-dimensional lattice with periodic boundary conditions and Hamiltonian

$$\beta H = -\sum_i^N J_i \sigma_i \sigma_{i+1} + h_i \sigma_i, \tag{2}$$

where β is the inverse temperature, J_i are nearest-neighbor couplings, and h_i are external fields (for instance, independent identically distributed random variables). The spins σ_i are dichotomic variables which can take the values ± 1 with equal probability. The partition function for a given realization of J's and h's is $Z_N = 2^N \langle \exp(-\beta H) \rangle_\sigma$, where the average $\langle\,\rangle_\sigma$ is performed over the 2^N spin configurations. In fact, it can be shown that the free energy for almost all realizations is given by the quenched average over the disorder:

$$-\beta F = \lim_{N \to \infty} \frac{1}{N} \overline{\ln Z_N} = \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr} \mathbf{G}_N = \lambda_1 \tag{3}$$

since Z_N is the trace of the product $\mathbf{G}_N$ of random transfer matrices of the form

$$\mathbf{T}(i) = \begin{pmatrix} \exp(J_i + h_i) & \exp(-J_i - h_i) \\ \exp(-J_i + h_i) & \exp(J_i - h_i) \end{pmatrix}. \tag{4}$$

In the following, the elements of $\mathbf{T}$ are indicated by $\exp(J_i \sigma_i \sigma_{i+1} + h_i \sigma_i)$ with obvious notation. The calculation of the quenched free energy of the random Ising model is thus equivalent to that of the maximum Lyapunov exponent λ_1. In a similar way, one has that the difference of the two Lyapunov exponents controls the decay rate $\langle \sigma_i \sigma_{i+r} \rangle \sim \exp[-(\lambda_1 - \lambda_2) r]$, for almost all realizations.

Let us briefly summarize the steps of our method.

(1) The initial couplings $\{J_1, J_2, \ldots, J_N\}$ and fields $\{h_1, h_2, \ldots, h_N\}$ are mapped into new ones, so that

$$\prod_i^N \mathbf{T}(i) = \exp\left(\sum_i^N \Gamma_i\right) \prod_i^N \mathbf{T}'(i), \tag{5}$$

where $\mathbf{T}'(i)$ is a transfer matrix with new couplings and fields h_i' and J_i'. One has that h_i' and J_i' and Γ_i are functions only of the old couplings and fields on the sites $i-1$, i, $i+1$.

(2) We iterate the mapping. It has an attractive fixed point at $h_i=0$ and $J_i=J^*$ for all sites i, where J^* depends on the starting h's and J's.

(3) The solution is given by

$$\prod_i^N \mathbf{T}(i) = \exp\left(\sum_{m=1}^{\infty} \sum_{i=1}^{N} \Gamma_i^{(m)}\right) \begin{pmatrix} \exp(J^*) & \exp(-J^*) \\ \exp(-J^*) & \exp(J^*) \end{pmatrix}^N, \tag{6}$$

where $\Gamma_i^{(m)}$ is obtained after m iterations of the map and depends on the starting couplings and fields on the $2m+1$ sites $i-m, \ldots, i+m$. The series in (6) is exponentially convergent. Note that up to this point, the method can be applied in the same way to find the solution for constant, quasiperiodic, or random fields and couplings.

(4) The most interesting case is fields and couplings which are random variables extracted according to a probability distribution $\rho(h,J)$. In the thermodynamic limit $N \to \infty$, almost all starting realizations converge to the same J^*. Moreover, the sums over the sites are given by disorder averages for self-averaging quantities like the free energy, i.e., $\lim_{N\to\infty} \sum_i \Gamma_i^{(m)}/N = \overline{\Gamma^{(m)}}$ for all i's.

The main trick of the method is to introduce a second set of spin variables η_i which can assume values of ± 1 with equal probability. This allows us to write the bilinear form $\sigma_i \sigma_{i+1}$ in $\mathbf{T}(i)$ as a form which is linear in σ_i and η_i. Indeed, the elements of $\mathbf{T}(i)$ satisfy the identity

$$\exp(J_i \sigma_i \sigma_{i+1} + h_i \sigma_i) = \langle \exp[A_i \eta_i (\sigma_i + \sigma_{i+1}) + B_i + h_i \sigma_i] \rangle_\eta, \tag{7}$$

where $\langle\ \rangle_\eta$ indicates the η average. Namely, after the average, the right-hand side of (7) becomes

$$\cosh[A_i(\sigma_i + \sigma_{i+1})] \exp(B_i + h_i \sigma_i) \tag{8}$$

leading to two independent equations (for instance, putting $\sigma_i = 1$ and $\sigma_{i+1} = \pm 1$) with the solution

$$B_i = -J_i, \quad A_i = \tfrac{1}{2} \cosh^{-1}(e^{2J_i}). \tag{9}$$

By means of (7) we compute the "thermal" σ average

$$\langle e^{-\beta H} \rangle_\sigma = \left\langle \left\langle \prod_i^N \exp[A_i \eta_i (\sigma_i + \sigma_{i+1}) + B_i + h_i \sigma_i] \right\rangle_\eta \right\rangle_\sigma = \prod_i^N e^{B_i} \langle \cosh(A_i \eta_i + A_{i-1} \eta_{i-1} + h_i) \rangle_\eta. \tag{10}$$

We can now return from (10) to a product of matrices $\mathbf{T}'$ of form (4), since it is always possible to find a set of constants Γ_i, H_i, and K_i which satisfy the identity

$$e^{B_i} \cosh(A_i \eta_i + A_{i-1} \eta_{i-1} + h_i) = e^{\Gamma_i + H_i(\eta_i + \eta_{i-1}) + K_i \eta_i \eta_{i-1}}. \tag{11}$$

Inserting (11) into (10), it follows that

$$\langle e^{-\beta H} \rangle_\sigma = \exp\left(\sum_i^N \Gamma_i\right) \left\langle \prod_i^N \exp(J_i' \eta_i \eta_{i+1} + h_i' \eta_i) \right\rangle_\eta, \tag{12}$$

where $\exp(J_i' \eta_i \eta_{i+1} + h_i' \eta_i)$ are the elements of $\mathbf{T}'(i)$ and the new fields and couplings $h_i' \equiv H_i + H_{i+1}$, $J_i' \equiv K_{i+1}$, as well as Γ_i are given by a mapping which can be obtained from (11). After some lengthy but trivial algebra, one finds

$$h_i^{(m+1)} = \langle x \ln[\cosh(xA_i^{(m)} + yA_{i+1}^{(m)} + h_{i+1}^{(m)}) \cosh(xA_i^{(m)} + yA_{i-1}^{(m)} + h_i^{(m)})] \rangle_{x,y},$$
$$J_i^{(m+1)} = \langle xy \ln\cosh(xA_i^{(m)} + yA_{i+1}^{(m)} + h_{i+1}^{(m)}) \rangle_{x,y}, \tag{13}$$

where m is the iteration step and $\cosh(2A_i^{(m)}) = \exp(2J_i^{(m)})$. The averages over two auxiliary random variables x and y which take value $=\pm 1$ with equal probability are introduced just in order to get a more compact form of the expressions. Note that (13) is a deterministic map with initial conditions $J_i^{(0)} = J_i$, $h_i^{(0)} = h_i$ and, at each step, the new fields and couplings depend only on the old ones on the nearest-neighbor sites. From (12) we can also derive the relation which gives Γ as

$$\Gamma_i^{(m+1)} = -J_i^{(m)} + \langle \ln\cosh(xA_i^{(m)} + yA_{i+1}^{(m)} + h_{i+1}^{(m)}) \rangle_{x,y}. \tag{14}$$

Map (13) has a line of fixed points at $h=0$. In fact, for a given disorder distribution $\rho(h,J)$ when $N \to \infty$, almost all initial realizations of $\{J_i\}$ and $\{h_i\}$ converge under the mapping to the same fixed point on this line, say, $h_i = 0$ and $J_i = J^*$, for all the lattice sites i. As an example, the flow of the averaged field and coupling for a random field Ising model is shown in Fig. 1.

This is the second key point of our discussion. Indeed, it solves the problem by reducing it to (6), i.e., to the diagonalization of the matrix with elements $e^{J^* \sigma_i \sigma_{i+1}}$ (whose eigenvalues are $2\cosh J^*$ and $2\sinh J^*$) plus the calculation of $\overline{\Gamma^{(m)}}$ along the map flow ($m = 1, 2, \ldots, \infty$). Therefore the two Lyapunov exponents are

$$\lambda_1 = \sum_{m=1}^{\infty} \overline{\Gamma^{(m)}} + \ln(2\cosh J^*),$$
$$\lambda_2 = \sum_{m=1}^{\infty} \overline{\Gamma^{(m)}} + \ln(2\sinh J^*), \tag{15}$$

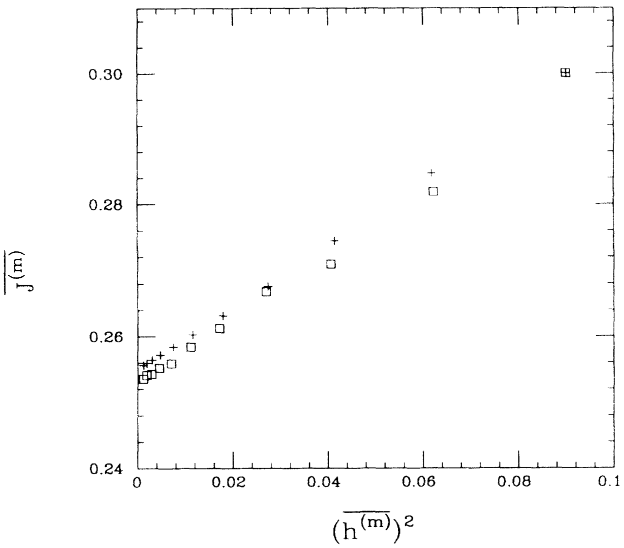

FIG. 1. Random field Ising model with constant $J=0.3$ and $h=0.2$ or $h=0.4$ with equal probability. Flow of $[\overline{h^{(m)}}]^2$, $\overline{J^{(m)}}$ for $m=0,1,\ldots,10$ (squares). The fixed point is $h^*=0$, $J^*=0.253$. For comparison, there is also the corresponding flow for the Ising model with constant field and coupling $J=0.3$, $h=0.3$ (crosses).

implying $J^*=\tanh^{-1}\exp[(\lambda_2-\lambda_1)/2]$.

We still have to prove that the fixed point is attractive. To do so, we linearize the first equation of (13) in the neighbor of the fixed point, i.e., for $h_i \sim 0$ and $J_i \sim J^*$, obtaining

$$h_i^{(m+1)}=\tfrac{1}{2}\tanh(2A^*)(h_i^{(m)}+h_{i+1}^{(m)}) \tag{16}$$

with $A^*=\frac{1}{2}\cosh^{-1}\exp(2J^*)$. Equation (16) implies an exponential convergence of $\overline{h^{(m)}}$ to zero whenever the two Lyapunov exponents are not equal, since the limit for $m\to\infty$ of

$$C_m \equiv \overline{h_i^{(m+1)}}/\overline{h_i^{(m)}}$$

is

$$C^*=\tanh(2A^*)=2e^{(\lambda_2-\lambda_1)/2}/e^{\lambda_2-\lambda_1}+1\,, \tag{17}$$

and the Perron theorem ensures that $\lambda_1 \neq \lambda_2$ at nonzero temperature. C_m also controls the flow of J and Γ in the neighbor of J^* and $\Gamma^*=0$ since by a Taylor expansion of (13) and (14) up to $O(h^2)$ one has

$$J_i^{(m+1)}-J_i^{(m)} \approx -\tfrac{1}{4}(1-e^{-4J^*})(h_{i+1}^{(m)})^2\,, \tag{18a}$$

$$\Gamma_i^{(m+1)} \approx \tfrac{1}{4}(1+e^{-4J^*})(h_{i+1}^{(m)})^2\,. \tag{18b}$$

As a consequence of (18a) and (18b), J^m and $\Gamma^{(m)}$ converge to their asymptotic limits much faster than $h^{(m)}$ (the h convergence rate $\ln C^*$ is half of the J and Γ convergence rates). This is illustrated by Fig. 1 where we have plotted the flow of $\overline{J^{(m)}}$ and $[\overline{h^{(m)}}]^2$, instead of $\overline{h^{(m)}}$. Let us stress the physical significance of the fixed point.

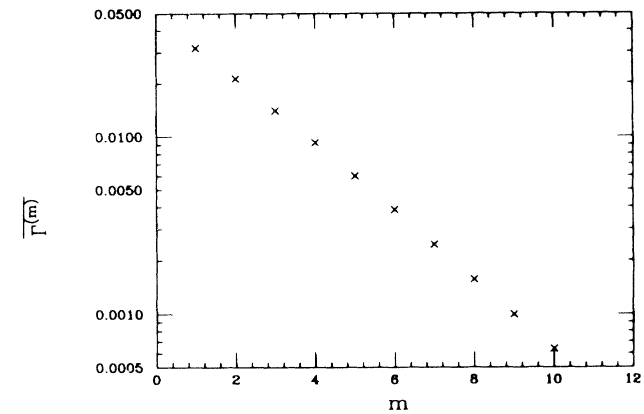

FIG. 2. $\overline{\Gamma^{(m)}}$ vs m using a log-linear scale for the random field Ising model with the same parameters of Fig. 1.

Under the mapping, both a random and a pure system are transformed into an equivalent Ising model at a temperature $\sim 1/J^*$ and zero magnetic field. Such a model has a very trivial solution, and all the difficulties are in the calculation of appropriate quantities along the trajectory of the system toward the fixed point. However, in a random system it is sufficient to follow the flow of the average coupling and field, since one is interested in the quenched free energy which is a self-averaging quantity. Moreover, an explicit calculation gives that $C_m=C^*+O(\exp(-Am))$. For practical purposes, this means that after calculating few k terms of the sum in (15), its remainder can be estimated by a geometrical series since

$$(\overline{h^{(k+m)}})^2 \approx (\overline{h^{(k)}})^2 C^{*2m}$$

apart from corrections which are exponentially small with $k+m$. One thus gets

$$\sum_{i=1}^{\infty}\overline{\Gamma^{(m)}} \approx \sum_{i=1}^{k}\overline{\Gamma^{(m)}}+(\overline{h^{(k)}})^2\left(\frac{1+e^{-4J^*}}{4}\right)\frac{1}{1-C^{*2}} \tag{19}$$

and a similar expression holds for J^*. Figure 2 shows the values of $\overline{\Gamma^{(m)}}$ as a function of m for the random field Ising model with the same parameters of Fig. 1. The linear shape of $\ln\overline{\Gamma^{(m)}}$ as a function of m provides evidence that the approximation (19) is already rather sensible for small k. In this case, a numerical calculation which performs the product of 10^6 random matrices gives $\lambda_1=0.8184$ and $\lambda_2=-0.5768$ with an estimated error of 10^{-4}. The results obtained by the asymptotic extrapolation (19) of the series truncated at $m=10$ are, respectively, $\lambda_1=0.8177$ and $\lambda_2=-0.5775$ (where $\overline{h^{(10)}}=3.42\ldots\times 10^{-2}$, $J^*=0.253\ldots$, and $C^*=0.798\ldots$).

It is worth stressing that (17) implies degenerate Lyapunov exponents if $J^*=\infty$, indicating that the one-dimensional random Ising model might exhibit a phase transition at $\beta=\infty$. In fact, map (13) also works at zero temperature after rescaling the variables $J_i\to J_i/\beta$, $h_i\to h_i/\beta$, and taking the limit $\beta\to\infty$. We can thus obtain the solution for the zero-temperature free energy $F = -\lim_{\beta\to\infty}\lambda_1(\beta)/\beta$ and decide whether a phase transition is present or not for a given disorder distribution $\rho(h,J)$. For instance, the random field Ising model can be shown to have no phase transition in the presence of strong disorder, i.e., probability distributions of the field such that $|h_i|/\beta > 2J/\beta$.

In conclusion, we have found a formal solution of the random Ising model in terms of an exponentially convergent series. The method used is reminiscent of the renormalization group (RG) in real space, *à la* Migdal and Kadanoff, as our mapping transforms the original system into a new one where the spin interactions are decreased (in average). However, a RG approach uses block variables so that the number of spins decreases at each transformation step, and in one dimension the fixed points correspond to zero or infinite temperature (that is, $J=\infty$, or $J=0$). On the other hand, the number of spins N remains constant under our mapping, which has a nontrivial fixed point corresponding to an Ising model of N spins at finite temperature $\sim 1/J^*$ and zero external field.

The speed of convergence toward the fixed point is determined by the difference of the two Lyapunov exponents, which is the typical decay rate of the two-point correlation functions. These considerations suggest that our method has a general validity which can be useful in the study of different one-dimensional random systems. Indeed, it should be remarked that our results have a straightforward application to any ensemble of random matrices with nonzero entries since they can always be written as matrices of form (4), while the case of matrices with zero entries—which includes important phenomena like the Anderson model [2]—is still far from being solved in our scheme. Moreover, the method can be extended to random Ising models on a strip of width L by considering $2^L \times 2^L$ transfer matrices, even if finding the appropriate mapping equations is not trivial as L increases. However, it is not difficult to find the mapping of the $L=2$ strip with zero magnetic field and random coupling which exhibits frustration phenomena, absent in one dimension [9]. This could be a first step toward an analytic approach for two-dimensional disordered systems.

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