Nonlinear Evolutions of Surface Gravity Waves on Fluid of Finite Depth

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New types of nonlinear evolution equations are derived that describe finite-amplitude surface gravity waves on a two-dimensional incompressible and inviscid fluid of finite depth. The novelty is as follows: (a) The equations can be expressed as a single equation with respect to the surface elevation, and (b) the expansion is in a steepness parameter and does not involve other approximations such as long waves or shallow water. Both the shallow- and deep-water limits of the equations are discussed.

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The study of finite-amplitude surface gravity waves has a long history [1,2]. Various wave phenomena, such as modulation effects and instabilities of wave trains [3], formation of solitary waves [4] and breaking waves [5], etc., have been investigated extensively from both the experimental and theoretical points of view. Many attempts have been made to derive approximate nonlinear evolution equations (NEEs) according to the physical situation. In the case of shallow-water waves, the Korteweg–de Vries (KdV) and Boussinesq equations were derived as typical model equations that describe the time evolution of the free-surface profile. For the finite-depth case, however, the corresponding equations have not been obtained explicitly as yet. In this respect a NEE proposed by Byatt-Smith should be remarked upon [6]. He derived an integrodifferential evolution equation for unsteady inviscid surface waves with the aid of the theory of conformal mappings. Although his equation is exact, it is quite complicated and it seems to be intractable without introducing some approximations. As for other related works, a review paper due to Miles [4] may be referred to.

The purpose of this Letter is to derive finite-depth analogs of the Boussinesq-type equations. A new method developed here is based on an elementary theory of complex functions and a systematic perturbation theory with respect to the steepness parameter.

We consider the two-dimensional irrotational flow of an incompressible and inviscid fluid of uniform depth. In the dimensionless variables, the equation governing the fluid motion is written in the form [1]

$$\delta^2\phi_{xx}+\phi_{yy}=0 \quad (-\infty<x<\infty,\quad -1<y<\alpha\eta)\,, \tag{1}$$

with the boundary conditions

$$\eta_t+\kappa\varepsilon\phi_x\eta_x=\frac{\kappa}{\delta}\phi_y \quad \text{on } y=\alpha\eta\,, \tag{2}$$

$$\phi_t+\frac{\kappa\varepsilon}{2\delta^2}(\delta^2\phi_x^2+\phi_y^2)+\eta=0 \quad \text{on } y=\alpha\eta\,, \tag{3}$$

$$\phi_y\to 0 \quad \text{as } y\to -1\,. \tag{4}$$

Here $\phi=\phi(x,y,t)$ is the velocity potential, $\eta=\eta(x,t)$ is the profile of the free surface, and the subscripts x, y, and t appended to ϕ and η denote partial differentiations. This notation will be used throughout the paper. η is assumed to be negative without loss of generality. The dimensional quantities, with tildes, are related to the corresponding dimensionless ones by the relations $\tilde{x}=lx$, $\tilde{y}=h_0y$, $\tilde{t}=(l/c_0)t$, $\tilde{\eta}=a\eta$, and $\tilde{\phi}=(gla/c_0)\phi$, where l, a, and c_0 are characteristic scales of length, amplitude, and velocity of the wave, respectively, $-h_0$ is the vertical coordinate of the flat bottom, and g is the acceleration due to gravity. The dimensionless parameters ε, α, and δ are then defined by $\varepsilon=a/l$, $\alpha=a/h_0$, and $\delta=h_0/l$. Note the relation $\varepsilon=\alpha\delta$. ε is called the steepness parameter. c_0 is given by $c_0=\sqrt{gl/\kappa}$ where κ is assumed to be δ^{-1} in the shallow-water limit $\delta\to 0$ and 1 in the deep-water limit $\delta\to\infty$ in accordance with the phase velocity of linear surface gravity waves. The effect of the surface tension has been neglected in (3) to simplify the discussion. It can be included without any difficulty.

Let us proceed to derive the time evolutions of the free surface and the horizontal component of the surface velocity. We first take the solution of (1) of the form

$$\phi=-i[f_+(x-i\delta y,t)-f_-(x+i\delta y,t)]\,, \tag{5}$$

where $f_+(z,t)$ $[f_-(z,t)]$ is an analytic function of z in the strip $0<\operatorname{Im}z<2\delta$ $(-2\delta<\operatorname{Im}z<0)$ and given explicitly by

$$f_\pm(z,t)=\pm\frac{1}{4i\delta}\int_{-\infty}^{\infty}\coth[\pi(y-z)/2\delta]f(y,t)dy\,. \tag{6}$$

Here f is an arbitrary real function defined appropriately on the real axis. The boundary condition (4) on the bottom of fluid is then found to be satisfied automatically. The key idea in the next step is to consider the boundary values of $f_\pm$ when $\operatorname{Im}z\to\pm 0$. It then turns out that

$$f_\pm(x\pm i0,t)=\tfrac{1}{2}(1\mp iT)f(x,t)\,, \tag{7}$$

where T is an integral operator given by

$$Tf(x,t)=\frac{1}{2\delta}P\int_{-\infty}^{\infty}\coth[\pi(y-x)/2\delta]f(y,t)dy\,. \tag{8}$$

The symbol P means the Cauchy principal-value integral.

By virtue of (7) we obtain the important relations

$$f_+(x+i0,t)+f_-(x-i0,t)=f(x,t)\,, \tag{9}$$

$$f_+(x+i0,t)-f_-(x-i0,t)=-iTf(x,t)\,. \tag{10}$$

On the free surface the derivatives of the velocity potential are evaluated from (5) as

$$\phi_x|_{y=a\eta}=-i[f_{+,x}(x-i\varepsilon\eta,t)-f_{-,x}(x+i\varepsilon\eta,t)]\,, \tag{11}$$

$$\phi_y|_{y=a\eta}=-\delta[f_{+,x}(x-i\varepsilon\eta,t)+f_{-,x}(x+i\varepsilon\eta,t)]\,, \tag{12}$$

$$\phi_t|_{y=a\eta}=-i[f_{+,t}(x-i\varepsilon\eta,t)-f_{-,t}(x+i\varepsilon\eta,t)]\,, \tag{13}$$

where the relation $\varepsilon=a\delta$ has been used. If we substitute (11)-(13) into (2) and (3), we obtain the *exact* system of NEEs for η and f.

In order to derive the approximate NEEs, we note that in the case of fluid of finite depth the parameters δ and κ may be assumed to be of the order of unity, whereas ε is small compared with unity. The latter assumption corresponds to considering small- but finite-amplitude waves and it enables us to expand (11)-(13) in powers of ε. In the following we derive NEEs correct up to $O(\varepsilon)$. The extension of the equations to higher orders can be made straightforwardly and hence all details are omitted.

If we expand (11)-(13) in ε and use (9) and (10), we immediately have the expressions for the first two terms of the expansions:

$$\phi_x|_{y=a\eta}=-Tf_x-\varepsilon\eta f_{xx}+O(\varepsilon^2)\,, \tag{14}$$

$$\phi_y|_{y=a\eta}=-\delta[f_x-\varepsilon\eta Tf_{xx}+O(\varepsilon^2)]\,, \tag{15}$$

$$\phi_t|_{y=a\eta}=-Tf_t-\varepsilon\eta f_{xt}+O(\varepsilon^2)\,. \tag{16}$$

At this stage it is convenient to introduce the horizontal component of the surface velocity given by $u=\phi_x|_{y=a\eta}$. Then f_x in (14) can be solved iteratively and it is expressed in terms of u as

$$f_x=-\tilde{T}u+\varepsilon\tilde{T}(\eta\tilde{T}u_x)+O(\varepsilon^2)\,, \tag{17}$$

where $\tilde{T}$ is an inverse operator of T, i.e., $\tilde{T}T=T\tilde{T}=I$ and it is given by

$$\tilde{T}u(x,t)=-\frac{1}{2\delta}P\int_{-\infty}^{\infty}\frac{u(y,t)}{\sinh[\pi(y-x)/2\delta]}dy\,. \tag{18}$$

Substituting (14), (15), and (17) into (2), we find the time evolution of η:

$$\eta_t-\kappa\tilde{T}u+\kappa\varepsilon[(u\eta)_x+\tilde{T}(\eta\tilde{T}u_x)]+O(\varepsilon^2)=0\,. \tag{19}$$

On the other hand, if we substitute (14)-(17) into (3) and then differentiate the resultant equation with respect to x and use the approximate equations $\eta_t=\kappa\tilde{T}u+O(\varepsilon)$ and $u_t=-\eta_x+O(\varepsilon)$ to eliminate time derivatives in $O(\varepsilon)$ terms, we obtain the time evolution of u:

$$u_t+\eta_x+\varepsilon[\kappa uu_x-\eta_x\tilde{T}\eta_x]+O(\varepsilon^2)=0\,. \tag{20}$$

The system of Eqs. (19) and (20) is just a finite-depth analog of the Boussinesq-type equations [1] in the theory of shallow-water waves. It is also possible to derive a single equation for η by combining (19) and (20). To show this we first solve (19) with respect to u iteratively to yield

$$\kappa u=T\eta_t+\varepsilon T[(\eta T\eta_t)_x+\tilde{T}(\eta\eta_{xt})]+O(\varepsilon^2)\,. \tag{21}$$

By substituting (21) into (20) and then operating $\tilde{T}$ on both sides, the desired equation for η arises as

$$\eta_{tt}+\kappa\tilde{T}\eta_x+\varepsilon[-\kappa\eta\eta_x+2\eta_tT\eta_t+\tilde{T}\eta_t^2-\kappa\tilde{T}(\eta\tilde{T}\eta_x)]_x+O(\varepsilon^2)=0\,, \tag{22}$$

where the approximate equation $\eta_{tt}=-\kappa\tilde{T}\eta_x+O(\varepsilon)$ has been used to eliminate η_{tt} in the $O(\varepsilon)$ term together with the formulas

$$\tilde{T}Tf=f,\quad \tilde{T}(fg)=\tilde{T}[(\tilde{T}f)(\tilde{T}g)]+g\tilde{T}f+f\tilde{T}g\,.$$

Equation (22) is a finite-depth analog of the Boussinesq equation. The new types of equations (19), (20), and (22) are the main results of the present Letter. In the linear approximation Eq. (22) reduces to $\eta_{tt}+\kappa\tilde{T}\eta_x=0$ and it exhibits a solution of the form $\eta=\eta_0\cos(kx-\omega t)$ with $\omega=\pm\sqrt{\kappa k}\tanh(\delta k)$, reproducing the exact linear dispersion relation for surface gravity waves on fluid of finite depth.

Next we discuss both the shallow- and deep-water limits of these equations.

(i) Shallow-water limit ($\delta\to 0$).—In this case we take $\kappa=\delta^{-1}$ and assume a to be small. With the expansion $\tilde{T}f=-\delta f_x-(\delta^3/3)f_{xxx}+O(\delta^5)$, Eqs. (19) and (20) reduce, respectively, to

$$\eta_t+u_x+a(u\eta)_x+\frac{\delta^2}{3}u_{xxx}+O(a\delta^2,\delta^4)=0\,, \tag{23}$$

$$u_t+\eta_x+auu_x+O(a\delta^2)=0\,. \tag{24}$$

The above system of equations is a variant of the Boussinesq system and is called the Broer-Kaup system [7-9]. This system has been shown to be completely integrable [7,9]. The usual form of the Boussinesq system [1] can be derived from (23) and (24) by introducing the mean horizontal velocity $\bar{u}$ which may be related to u by $u=\bar{u}-(\delta^2/3)\bar{u}_{xx}$ and assuming $a=O(\delta^2)$. For the purpose of taking the shallow-water limit of Eq. (22), we employ the expansion $Tf=(2\delta)^{-1}\int_{-\infty}^{\infty}\mathrm{sgn}(y-x)f(y,t)dy+O(\delta)$ where sgn$x$ is the sign function. This leads to the equation

$$\eta_{tt}-\eta_{xx}-\frac{\delta^2}{3}\eta_{xxxx}-a\left((\eta\eta_x)_x+2\eta_t^2-\eta_{xt}\int_{-\infty}^{\infty}\mathrm{sgn}(y-x)\eta_t(y,t)dy\right)+O(a\delta^2,\delta^4)=0\,. \tag{25}$$

The Boussinesq equation,

$$\eta_{tt} - \eta_{xx} - \frac{\delta^2}{3}\eta_{xxxx} - 3\alpha(\eta\eta_x)_x + O(\alpha\delta^2,\delta^4) = 0 , \quad (26)$$

can be obtained from (25) with the assumptions that (a) $\alpha = O(\delta^2)$ and (b) η_t may be replaced by $-\eta_x$ in $O(\alpha,\delta^2)$ terms which represent nonlinearity and dispersion. However, since Eq. (26) admits both right- and left-moving solutions, only the former solutions are consistent with the assumption (b). It should be remembered, however, that the original Boussinesq equation (26) rests on these assumptions [4]. From this point of view Eq. (25), which does not depend on (b), is superior to Eq. (26). One notes that Byatt-Smith [6] also obtained a similar equation to (25). To derive an equation which describes a unidirectional motion to the right, for instance, it is appropriate to use a reference frame moving with the phase velocity of the wave. In this system wave profiles change very slowly so that it is legitimate to introduce a slowly varying time scale. The corresponding coordinate and time transformations may be expressed as $\xi = x - t$ and $\tau = \alpha t/2$ since in the present case the phase velocity has been normalized to unity. Then Eq. (25) reduces, under the assumption of $\alpha = O(\delta^2)$, to

$$\eta_\tau + 3\eta\eta_\xi + \frac{\delta^2}{3\alpha}\eta_{\xi\xi\xi} + O(\alpha,\delta^2) = 0 , \quad (27)$$

which is nothing but the KdV equation.

(ii) Deep-water limit ($\delta \to \infty$).—The deep-water limit can be taken quite easily because in this limit the operators T and $\tilde{T}$ reduce to H and $-H$, respectively, where H is the Hilbert transform, i.e., $Hf(x) = (1/\pi)P\int_{-\infty}^{\infty}(y-x)^{-1}f(y)dy$. In the basic equations (1)-(4), however, it is necessary to rescale the vertical coordinate as $\hat{y} = \delta y$ before taking the limit. As a result only one parameter ε exists in the system under consideration when $\delta \to \infty$ since $\kappa = 1$ in this limit. Equations (19), (20), and (22) then become

$$\eta_t + Hu + \varepsilon[(u\eta)_x + H(\eta H u_x)] + O(\varepsilon^2) = 0 , \quad (28)$$

$$u_t + \eta_x + \varepsilon[uu_x + \eta_x H\eta_x] + O(\varepsilon^2) = 0 , \quad (29)$$

$$\eta_{tt} - H\eta_x - \varepsilon[\eta\eta_x + H(\eta H\eta_x) + H(H\eta_t)^2]_x + O(\varepsilon^2) = 0 , \quad (30)$$

where use has been made of the formula $H(\eta_t)^2 = H(H\eta_t)^2 + 2\eta_t H\eta_t$ in deriving (30). Equation (30) exhibits solutions which propagate in both directions as indicated by the linear dispersion relation $\omega = \pm\sqrt{|k|}$. However, we have not succeeded as yet in obtaining the equation corresponding to the KdV equation. The situation is the same as that for Eq. (22) and these interesting problems should be pursued further.

In this Letter we have derived new types of equations, in particular Eqs. (19), (20), (22), (25), and (28)-(30), that describe finite-amplitude surface gravity waves by means of a systematic perturbation method. In future work we shall reexamine various wave phenomena mentioned in the introductory part on the basis of these equations.

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