Energy Spectra in a Model for Convective Turbulence

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The energy cascade in both hydrodynamic and hydromagnetic Boussinesq convection is investigated at large Rayleigh numbers, using a scalar model for turbulence. Depending on the relative importance of direct and inverse transfer, either we observe classical Kolmogorov $k^{-5/3}$ spectra or, if there is a strong inverse transfer of kinetic energy, we find a $k^{-7/5}$ spectrum for the temperature fluctuation and a $k^{-11/5}$ spectrum for the kinetic energy (Bolgiano-Obukhov scaling). We derive dissipative cutoff wave numbers that are consistent with these spectra.

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Recent experiments on convection at high Rayleigh numbers [1] have indicated that the frequency spectrum of the temperature fluctuation follows a power law with an exponent close to $-7/5$. Using the Taylor hypothesis this translates to a $k^{-7/5}$ wave-number spectrum, which significantly differs from the Kolmogorov $k^{-5/3}$ spectrum. This discovery has triggered a number of theoretical investigations. A $k^{-7/5}$ spectrum for the temperature fluctuation and a $k^{-11/5}$ spectrum for the kinetic energy has been suggested by Bolgiano [2] and Obukhov [3] for turbulence in a stably stratified medium. Procaccia and Zeitak [4] found the same scaling also for the contrasting case of convective turbulence, employing a dynamical theory for the structure functions. However, Shraiman and Siggia [5] argued that Bolgiano-Obukhov scaling is inconsistent in this case and that the wave-number spectrum might actually be steeper than the frequency spectrum. Moreover, Castaing [6] demonstrated that the experimental result could also be fitted by a $k^{-5/3}$ law if the crossover from the inertial to the dissipative subrange is included. Grossmann and Lohse [7] also presented analytical arguments and numerical simulations in favor of a Kolmogorov spectrum. A more recent experiment [8] seems again to confirm the $-7/5$ exponent, not only for the frequency but also directly for the wave-number spectrum of the temperature fluctuation.

In this Letter we study a simple cascade model for which we find both Kolmogorov and Bolgiano-Obukhov scaling, depending on the details of the energy transfer. This allows us to identify the physical mechanism responsible for the occurrence of one or the other possible scaling laws. In some cases we also include the effects of magnetic fields relevant in the astrophysical context.

The range of length scales currently resolved in direct simulations of three-dimensional turbulent hydrodynamic and hydromagnetic convection [9,10] hardly exceeds 2 orders of magnitude. Thus, inertial subranges for the energy spectra are short and the scaling exponents cannot be determined to very high precision. This particular disadvantage is avoided in a highly idealized, one-dimensional, scalar model for turbulence, where only interactions between wave numbers k from nearby shells are taken into account (therefore also known as a "shell model"). By choosing an exponential spacing in k one can easily cover many orders of magnitude, and thus very high Reynolds and Rayleigh numbers are possible. The nonlinear interactions in such models conserve certain quantities which are also conserved by the original equations.

Scalar models have been used to study, in the nonmagnetic case, cascade processes in turbulence [11], intermittency corrections from Kolmogorov scaling [12], and Obukhov-Corrsin scaling for the advection of a passive scalar [13]. In the magnetic case, properties of hydromagnetic turbulence and dynamo effect have been investigated [14,15]. In all these studies the flow has been externally forced by an energy input at small wave numbers.

Forcing terms do not occur in the original Boussinesq equations [16]. The flow results from the linear Rayleigh-Bénard instability once the temperature gradient between the top and the bottom of the fluid layer exceeds a certain threshold. In a previously presented cascade model for Boussinesq convection [7], where the boundary condition for the temperature difference was difficult to implement, an external forcing was assumed instead. Since the nature of this forcing might be crucial [17], we choose to work with the temperature fluctuation Θ, which vanishes on perfectly conducting boundaries, but leads to a new term in the Θ equation [16]. For large Rayleigh numbers many modes are excited by the linear instability and an external forcing becomes superfluous.

A related approach has been proposed by Legait [18], who considered a scalar model for turbulent convection using two coupled equations for the horizontal and vertical vorticity. However, the temperature does not explicitly enter and the buoyancy term is parametrized using an expression for the growth rate as a function of wave number. Since we are particularly interested in the spectrum for Θ we simultaneously solve model equations for Θ and for the velocity. A scalar is used to represent the collective behavior of the velocity in a certain wave-number band.

The set of equations governing hydromagnetic Boussinesq convection in a layer of thickness d are [9,10,16]

$$\left(\frac{\partial}{\partial t}-\chi\nabla^2\right)\Theta=-\mathbf{u}\cdot\nabla\Theta+\beta\mathbf{u}\cdot\hat{\mathbf{z}}\,, \tag{1}$$

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\mathbf{u}=-\nabla p'-\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}+\alpha g\Theta\hat{\mathbf{z}}\,, \tag{2}$$

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\mathbf{b}=-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u}\,, \tag{3}$$

with $\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{b}=0$. Here, $\mathbf{u}$ is the velocity, $\mathbf{b}$ is the magnetic field (normalized to have the same dimensions as velocity), p' is the pressure fluctuation (including the magnetic pressure) relative to the hydrostatic equilibrium state with constant temperature gradient β, α is the volume expansion coefficient, and g is gravity. χ, ν, and η denote thermal conductivity, kinematic viscosity, and magnetic diffusivity, respectively. To simplify the presentation we consider impenetrable, thermally and magnetically perfectly conducting boundaries and require

$$\Theta=u_{x,z}=u_{y,z}=u_z=b_{x,z}=b_{y,z}=b_z=0 \text{ on } z=0,d\,, \tag{4}$$

where commas denote derivatives.

In the inviscid case (1)-(4) lead to the following integral properties:

$$\frac{d}{dt}\int\tfrac{1}{2}(\mathbf{u}^2+\mathbf{b}^2)d^3x=\frac{\alpha g}{\beta}\frac{d}{dt}\int\tfrac{1}{2}\Theta^2d^3x$$
$$=\alpha g\int u_z\Theta\, d^3x \tag{5}$$

and

$$\frac{d}{dt}\int\mathbf{u}\cdot\mathbf{b}\,d^3x=\alpha g\int b_z\Theta\, d^3x\,. \tag{6}$$

Note that in this case the quantity

$$\int\tfrac{1}{2}[\mathbf{u}^2+\mathbf{b}^2-(\alpha g/\beta)\Theta^2]d^3x \tag{7}$$

is conserved.

We now introduce nondimensional variables by measuring length in units of d, time in units of $(\alpha\beta g)^{-1/2}$, and Θ in units of βd, i.e., we put $\alpha=\beta=g=1$. We define scalar variables u_n, b_n, and Θ_n on a one-dimensional k mesh representing different shells in wave-number space with

$$k_n=k_0h^n,\quad n=0,1,\dots,N-1\,, \tag{8}$$

where N is the total number of modes and k_0 is the wave number corresponding to the largest possible scale in the system. We put $k_0=1$ and, in accordance with previous scalar models, $h=2$. (For smaller values of h the spectra are somewhat smoother, but the slope remains unchanged.)

Analogously to (5) and (6) we require that

$$\frac{d}{dt}\sum\tfrac{1}{2}(u_n^2+b_n^2)=\frac{d}{dt}\sum\tfrac{1}{2}\Theta_n^2=\sum u_n\Theta_n \tag{9}$$

and

$$\frac{d}{dt}\sum u_nb_n=\sum b_n\Theta_n \tag{10}$$

are satisfied by the inviscid equations.

For the evolution of Θ_n we write

$$\left(\frac{d}{dt}+\chi k_n^2\right)\Theta_n=k_n\sum_{i,j=-1,0,1}c_{ij}u_{n+i}\Theta_{n+j}+u_n\,. \tag{11}$$

The coupling coefficients c_{ij} are determined such that (9) is satisfied. The equations for u_n and b_n can readily be adopted from Ref. [14], where only the buoyancy term Θ_n has to be included in the equation for u_n. The complete set of equations then reads

$$\left(\frac{d}{dt}+\chi k_n^2\right)\Theta_n=\tilde{A}k_n(u_{n-1}\Theta_{n-1}-hu_n\Theta_{n+1})+\tilde{B}k_n(u_n\Theta_{n-1}-hu_{n+1}\Theta_{n+1})+u_n\,, \tag{12}$$

$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n=Ak_n[u_{n-1}^2-b_{n-1}^2-h(u_nu_{n+1}-b_nb_{n+1})]+Bk_n[u_nu_{n-1}-b_nb_{n-1}-h(u_{n+1}^2-b_{n+1}^2)]+\Theta_n\,, \tag{13}$$

$$\left(\frac{d}{dt}+\eta k_n^2\right)b_n=Ak_nh(u_{n+1}b_n-u_nb_{n+1})+Bk_n(u_nb_{n-1}-u_{n-1}b_n)\,. \tag{14}$$

The coefficients A, B, $\tilde{A}$, and $\tilde{B}$ quantify the relative importance of inverse and direct transfer by the nonlinear terms; see Ref. [14] for a detailed discussion. For example, without magnetic fields and in the absence of the buoyancy term Θ_n in (13), the case $A=0$ only causes an inverse transfer toward smaller k via the term $-Bk_nhu_{n+1}^2$. At first glance, a direct transfer via the term $Bk_nu_nu_{n-1}$ is also possible. However, this term can only lead to an exponential growth of u_n, if $Bu_{n-1}>0$. Now, the other term $-Bk_nhu_{n+1}^2$ gives a systematically *negative* contribution to u_{n-1}, regardless of the sign of u_n. Thus, the energy at higher wave numbers tends to decay, which we also confirmed numerically.

Note that neither Bolgiano-Obukhov scaling nor Kolmogorov scaling corresponds to a fixed point of the inviscid equations. However, in the absence of the u_n term in (12), the Bolgiano-Obukhov scaling ($u_n\propto k_n^{-3/5}$, $\Theta_n\propto k_n^{-1/5}$, and $b_n=0$) would correspond to an unstable fixed point of the inviscid equations. If, in addition, the buoyancy term Θ_n in (13) were also absent then Kolmogorov scaling ($\Theta_n\propto u_n\propto b_n\propto k_n^{-1/3}$) would correspond to an unstable fixed point (cf. Ref. [14]). We note furthermore that only in the magnetic case does the system of equations satisfy the Liouville theorem [14].

By linearizing (12)-(14) we see that the state Θ_n

$=u_n=b_n=0$ becomes unstable at $\mathrm{Ra}\equiv\mathrm{Ra}_c=k_0^4\equiv 1$, where $\mathrm{Ra}=(\nu\chi)^{-1}$ corresponds to a Rayleigh number. The nonmagnetic state $b_n=0$ (but $\Theta_n,u_n\neq 0$) becomes linearly unstable (dynamo effect), if $(Ahu_{n+1}-Bu_{n-1})/\eta k_n>1$ for some n. For the remainder we focus on large Rayleigh numbers so that the solutions are chaotic. The results presented below are for $\chi=\nu=\eta$.

We numerically integrated (12)-(14) using as initial condition $\Theta_n=u_n=b_n=0$ with a small perturbation in u_n (and b_n) for an intermediate value of n. We investigated the solutions for various coefficients A, B, $\tilde{A}$, and $\tilde{B}$. Note that one of these four coefficients can always be normalized to unity. We mostly used $A=0.01$, $B=\tilde{A}=\tilde{B}=1$. (See Refs. [14,15] for a discussion of the case of small A.) We looked at the various time-averaged spectra $E_\Theta=\langle\Theta_n^2\rangle/2k_n$, $E_K=\langle u_n^2\rangle/2k_n$, $E_M=\langle b_n^2\rangle/2k_n$, and pushed the Rayleigh number as high as possible so that we just resolve a dissipative subrange. Using $N=30$ modes we were able to reach $\mathrm{Ra}/\mathrm{Ra}_c=2.5\times 10^{25}$ in the nonmagnetic case and 4×10^{22} in the magnetic case.

In the absence of magnetic fields we find an extended inertial subrange with $E_K\propto k^{-11/5}$ and $E_\Theta\propto k^{-7/5}$ (Bolgiano-Obukhov scaling); see Fig. 1. It is interesting to note that these spectra are obtained only after averaging over many hundred time units. The instantaneous energy of modes in the inertial range can vary over 6 orders of magnitude.

The Bolgiano-Obukhov scaling is closely related to the relative importance of inverse transfer of kinetic energy measured by the ratio $|B/A|$. If this ratio is below some critical value (around 0.4) we find classical Kolmogorov scaling. The qualitative behavior does not seem to be very sensitive to the exact values of $\tilde{A}$ and $\tilde{B}$, provided there is a direct transfer of either kinetic energy or temperature. (The case $|B/A|\gtrsim 1$ and $|\tilde{B}/\tilde{A}|\gtrsim 1$ leads to an unlimited growth of energy at large scales.)

In the presence of magnetic fields we find for all combinations of A, B, $\tilde{A}$, and $\tilde{B}$, Kolmogorov $k^{-5/3}$ spectra for E_Θ, E_K, and E_M; see Fig. 2. Note that the famous $k^{-3/2}$ spectrum [19] is only to be expected when nonlocal interactions via the Alfvén effect are taken into account [14].

From dimensional arguments one can see that the different scaling behaviors crucially depend on the role of the buoyancy term $\alpha g\Theta$ in determining the energy cascade [5,7]. If the coefficient αg is important for the cascade then the spectra must have the form

$$E_K=E_K(\alpha g,\epsilon_\Theta,k)=C_K(\alpha g)^{4/5}\epsilon_\Theta^{2/5}k^{-11/5}, \qquad (15)$$

$$E_\Theta=E_\Theta(\alpha g,\epsilon_\Theta,k)=C_\Theta(\alpha g)^{-2/5}\epsilon_\Theta^{4/5}k^{-7/5}, \qquad (16)$$

where $\epsilon_\Theta=\chi\sum k_n^2\Theta_n^2$ is the rate of Θ dissipation. (For the case depicted in Fig. 1 we obtain $C_K\approx 0.8$ and $C_\Theta\approx 2$.) On the other hand, if the cascade is governed by the kinetic-energy dissipation then

$$E_K=E_K(\epsilon_K,k)=C_K\epsilon_K^{2/3}k^{-5/3}, \qquad (17)$$

$$E_\Theta=E_\Theta(\epsilon_K,\epsilon_\Theta,k)=C_\Theta\epsilon_K^{-1/3}\epsilon_\Theta k^{-5/3}, \qquad (18)$$

$$E_M=E_M(\epsilon_K,\epsilon_M,k)=C_M\epsilon_K^{-1/3}\epsilon_M k^{-5/3}, \qquad (19)$$

where $\epsilon_K=\nu\sum k_n^2u_n^2$ is the rate of viscous dissipation and $\epsilon_M=\chi\sum k_n^2b_n^2$ is the rate of Joule dissipation. (For the case depicted in Fig. 2 we obtain $C_K\approx 2.5$, $C_\Theta\approx 1.5$, and $C_M\approx 3$.)

The dissipative cutoff wave numbers, k_K, k_Θ (and k_M), above which the dissipative subranges in E_K, E_Θ (and E_M) begin, can be estimated using (15)-(19) to solve $\epsilon_K=2\nu\int_0^{k_K}k^2E_K\,dk$, and similarly for k_Θ and k_M. This yields

$$k_K=(\tfrac{5}{2}\nu C_K)^{-5/4}(\alpha g)^{-1}\epsilon_\Theta^{-1/2}\epsilon_K^{5/4}, \qquad (20)$$

$$k_\Theta=(\tfrac{5}{4}\chi C_\Theta)^{-5/8}(\alpha g)^{1/4}\epsilon_\Theta^{1/8}. \qquad (21)$$

Inconsistencies have been noted [5] if k_Θ [see (11c) in

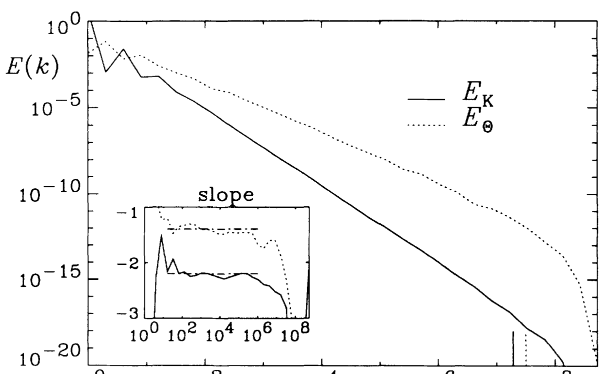

FIG. 1. Spectra for the nonmagnetic case with $\mathrm{Ra}/\mathrm{Ra}_c=4\times 10^{26}$. The inset shows that the local slope $d\ln E/d\ln k$ is around $-\frac{7}{5}$ for E_Θ and around $-\frac{11}{5}$ for E_K (horizontal dash-dotted lines). The two vertical bars on the k axis mark dissipative cutoff wave numbers.

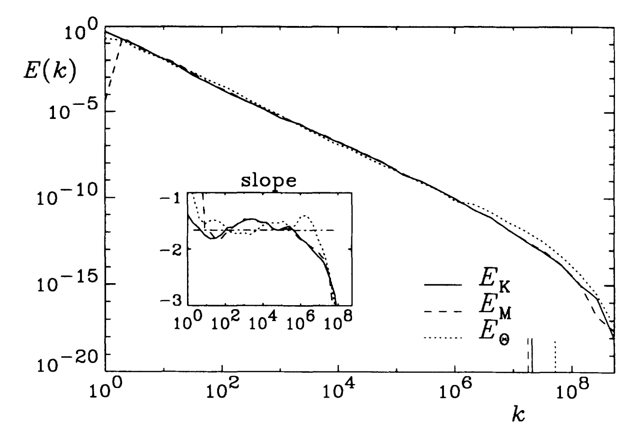

FIG. 2. Spectra for the magnetic case with $\mathrm{Ra}/\mathrm{Ra}_c=10^{24}$. The inset shows that the local slope for all three spectra varies around $-\frac{5}{3}$. The three vertical bars on the k axis mark dissipative cutoff wave numbers.

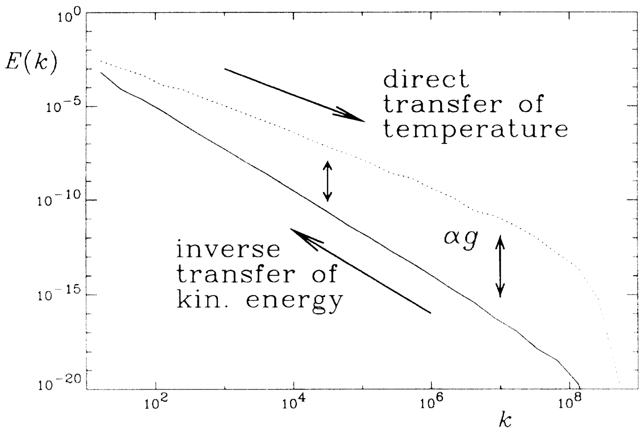

FIG. 3. Sketch of transfer properties in the case of Bolgiano-Obukhov scaling. The symbol αg refers to the exchange of energy via buoyancy work.

Ref. [5]] (instead of k_K) was used to estimate the kinetic-energy dissipation. Note that the use of Eq. (20) automatically avoids this problem. The cutoff for the spectra (17)-(19) has the well-known form $k_\Theta = [\epsilon_\Theta / (6\chi C_\Theta)^3]^{1/4}$ (and analogously for K and M).

In Fig. 3 we illustrate the transfer properties for which Bolgiano-Obukhov scaling can be expected. If, for some reason, an inverse transfer of kinetic energy is very strong then the energy will dissipate via Θ dissipation. Therefore the coupling coefficient αg, which regulates the interaction between u and Θ, becomes important in (15) and (16). However, in the magnetic case equipartition ($b_n \approx \pm u_n$) is rapidly achieved and a direct transfer via b_n becomes possible [14], leading to Kolmogorov scaling.

Because of the limitations of this scalar model we cannot claim that Bolgiano-Obukhov scaling will necessarily occur in nature. The aim of this Letter is rather to present evidence that Bolgiano-Obukhov scaling might be closely connected to a strong inverse transfer of kinetic energy. In fact, we find that Bolgiano-Obukhov scaling disappears for various modifications to this model if, for example, complex variables are used or if interactions between horizontal and vertical vorticity modes are included [18]. However, in these cases inverse transfers turn out to be weak. When the interactions with next-nearest neighbors are included according to Refs. [12,13] there is an unbounded growth of the velocity at small k and the instantaneous spectra are typically of the Kolmogorov type. However, if next-nearest-neighbor interactions are included using terms of the form $k_n(u_n u_{n-2} - h^2 u_{n+2}^2)$ we do obtain a statistically steady state with Bolgiano-Obukhov scaling. Thus, nonlocality in k space [20] can be crucial, depending on which kind of wave-number interactions are included. Indeed, the results presented here are model dependent and Bolgiano-Obukhov scaling is found only in special cases where inverse transfers are sufficiently strong and where the buoyancy term αg plays an important role for the energy transfer.

It is well known [21] that in two-dimensional turbulence strong inverse transfers result from pairing of large energetic eddies leading ultimately to a large-scale vortex occupying the entire system. This appears to be similar to the large scale flow observed in high-Rayleigh-number convection [22]. This flow corresponds to a large scale two-dimensional vortex. Although we expect convection to be truly three dimensional we thus do see that this cannot be the case at large scales. Therefore, the question whether or not Bolgiano-Obukhov scaling occurs seems to be intimately related to possible inverse transfers associated with this large scale flow.

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