Saffman-Taylor Plumes

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We present an experimental study of thermal plumes growing in Hele-Shaw cells which shows the existence of new objects, analogous to Saffman-Taylor fingers, for which the thermal boundary layer plays the role of the interface. The theoretical analysis reveals the underlying selection mechanism, which is provided by heat diffusion.

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These last years, much effort has been devoted to the study of interface dynamics problems, such as viscous fingering [1,2], crystal growth [3], flame propagation [4], and electrolytic deposition [5]; despite the diversity of the situations, the questions of pattern stability and selection have received uniform theoretical treatment, and some of them have been reasonably solved, at least in the simplest cases [2,3]. The purpose of this Letter is to show, for the first time, that a similar approach also applies for the problem of determining the shapes of thermal boundary layers. Although thermal plumes have been studied extensively in the past [6], interest in the subject was revived recently because of their relevance to turbulence [7] and geophysics [8]. However, here we regard these structures from the point of view of pattern formation.

The particular problem that we investigate presently is hot fluid displacing colder fluid in a vertical Hele-Shaw channel. The latter is formed by clamping together two 4-mm-thick Plexiglas plates, separated by Plexiglas spacers; channels of different aspect ratios have been studied, but most of the results presented herein have been obtained on a cell of width $w = 3.00 \pm 0.01$ cm, thickness $b = 2.35 \pm 0.05$ mm, and length 32 cm. A $\frac{1}{4}$-W resistance, placed inside the channel about 1 cm above the lower end, serves as a heater to produce the thermal plumes. Heater lengths considered in this study range from 1 to 3 cm. The whole ensemble is immersed vertically in a glass vessel filled with the working fluid. The ends of the Hele-Shaw cell are left open, in free contact with the fluid reservoir of the vessel. The fluids which we use are water and silicone oils. We visualize the rising warm front by means of a shadowgraph technique, employing a point source of white light, an 8 cm, $f = 50$ cm lens, and a TV camera.

If we turn on a localized heater inside a fluid initially at rest and far from any walls, the hot fluid rises in a column surmounted by a wider cap, forming a three-dimensional thermal plume; such an object resembles a mushroom and has an axisymmetric structure. With a constant power input at the heater, the cap rises at constant velocity, while its lateral size grows with the square root of time; its shape is well approximated, at all instants, by a Rankine fairing [9]. In a Hele-Shaw cell, provided the heater is small compared to the channel width, we initially form a thermal plume which expands rapidly, just like in the infinite-geometry case. At later times a different regime sets in, and the plume transforms into a "finger," similar to those formed by the interface between two immiscible fluids in a Hele-Shaw cell [1]. Figure 1 is a shadowgraph of such a finger, of width $\lambda \approx 0.56$ (relative to the channel size), moving at 1 cm/s; in the present series of experiments, the finger velocity is not strictly constant, presumably on account of the thermal losses, but the corresponding decrease is so slow that we can regard the plume as quasistationary. In Fig. 1, the boundary between the hot and cold fluid is the dark region just inside the bright line. Similarities with the Saffman-Taylor (ST) fingers are striking: The shape of the front agrees well with the solutions found by Saffman and Taylor [1], as shown in Fig. 2; in contrast, a fit with a Rankine shape (the form of the free plume) would be noticeably worse. Concerning the finger tail, we can see that it does not remain asymptotically parallel to the lateral walls, but smoothly bends towards the center; inside the finger there is a recirculating flow which maintains a sharp temperature drop at the "interface," that is, which sustains the boundary layer.

In Fig. 3, we present a plot of λ versus Peclet number $\mathrm{Pe} = Uw/\kappa$, where U is the velocity of the finger, w the width of the cell, and κ the thermal diffusivity (the intro-

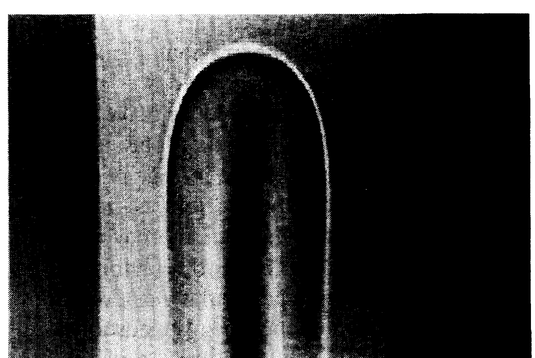

FIG. 1. Shadowgraph picture of a thermal plume growing in a Hele-Shaw cell 3 cm wide, 2.35 mm thick; Pe = 3160, $\lambda = 0.56$. The dark regions on the sides are the lateral walls of the cell.

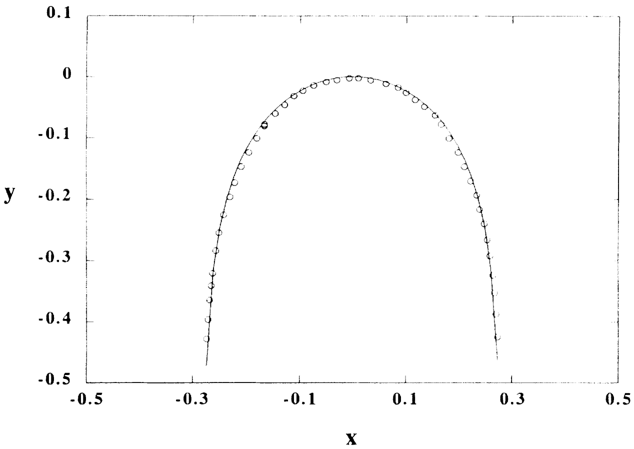

FIG. 2. Comparison between the plume of Fig. 1 (circles) and the Saffman-Taylor analytical solution (line) with the same λ.

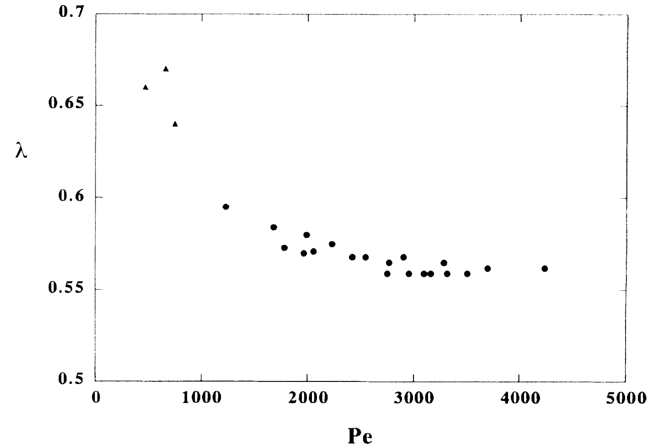

FIG. 3. Relative widths of plumes, plotted as a function of Peclet number, for two different oils, in the same channel ($b = 2.35$ mm and $w = 3$ cm). ▲: 20-cS oil; ●: 2-cS oil.

duction of this quantity will be justified later). The measurements of Fig. 3 are performed far from the heater, at distances of typically 6 cell widths from it, above which λ is found approximately constant. The points at the lowest Peclet numbers are obtained in 20-cS oil, and the others in 2-cS oil, the range of Peclet number being covered by varying U [1 centistoke (cS) $= 10^{-2}$ cm^2/s]. The dependence of λ with Peclet number is found to be small in any case. The asymptotic limit at large Peclet number, which is $\lambda \approx 0.55$, is slightly above the width of ST fingers; such a difference may be related to the finite thickness of the cell. Additional data, obtained in cells of different aspect ratios, tend to support this conclusion.

In order to establish more precisely the connection with the ST problem, we investigate the effect of time-independent perturbations on the thermal fingers. We find that the phenomenology is remarkably close to the one observed in viscous fingering [10]: Perturbations always lead to fingers with dramatically smaller sizes. We present two examples: In Fig. 4(a) the plume was started only 10 s after the heating to the preceding plume was switched off; the stem of the previous plume is still dimly visible in the picture. This provides a perturbation of both the initial temperature and velocity fields, and the result is a finger with $\lambda < \frac{1}{2}$ (this plume is obtained with the same power input as the one in Fig. 1). In Fig. 4(b) a 170-μm-thick metallic wire is stretched inside the channel. This again leads to $\lambda < \frac{1}{2}$; moreover, the finger is displaced asymmetrically inside the channel so as to accommodate the wire at a particular position along the interface: This phenomenon was also observed in viscous fingering experiments [10]. In the case of the plume, the particular position corresponds to the insertion of the stem onto the cap, as is evident from the picture.

We now consider the theoretical problem, for which we try to develop a strategy similar to the Saffman-Taylor problem. However, in the present case, the interface between the two fluids is diffuse, and, in order to make the analogy explicit, one needs to transform, in a consistent way, the thermal boundary layer into a sharp interface. We do this using the "phase field model" [11]; briefly, it consists in dividing the physical space into three regions: the diffuse interface, and the space on each side of it. The relevant fields are expanded in powers of a small pa-

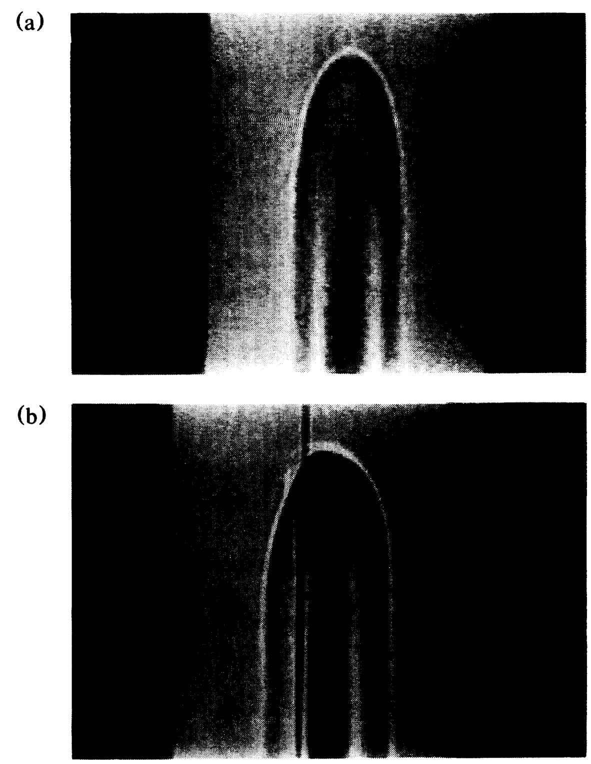

FIG. 4. Perturbed Saffman-Taylor plumes: (a) using the stem of a preceding plume ($\lambda = 0.39$) and (b) with a wire, 170 μm in diameter ($\lambda = 0.46$).

rameter (related to the interface thickness), and matched for the different regions. The ensuing solvability condition gives the boundary conditions on the interface. To apply this method, we introduce simplifying assumptions, which are suggested by the experiment. We assume that the system is two dimensional, the flow is governed by D'Arcy law (for the sake of simplicity, we take the viscosity constant), and we consider that the temperatures, far away from the boundary layer, are steady and homogeneous, with values T_1 inside the finger, and T_2 outside (with $T_1 > T_2$). We now seek steady-state solutions where the boundary layer propagates at constant velocity U; we introduce the stream function $\Psi(x,y)$ which, in a frame moving at velocity U, is related to the thermal field $\theta(x,y)$ by D'Arcy law with gravity, which reads [6]

$$\Delta\Psi = \alpha \frac{\partial\theta}{\partial x}, \tag{1}$$

where x,y are the coordinates (the y axis being directed along the mean flow), and $\alpha = (b^2/12\mu)g\rho\alpha'/U$ (here μ, ρ, and α' are respectively the viscosity, density, and thermal expansion coefficient of the fluid, and g is the acceleration of gravity). In (1), lengths, velocities, and temperatures are rescaled by using w, U, and $T_1 - T_2$. Note that in the experiments described above, we have $\alpha \sim 1$.

In the spirit of the phase field model, we further add to the advection diffusion equation a forcing term $F(\theta)$, whose role is to force the temperature $\theta(x,y)$ to have two distinct, constant values outside the boundary layer, 1 and -1, respectively, inside and outside the finger. The equation for $\theta(x,y)$ is then (in the frame moving with the finger)

$$\mathrm{Pe}^{-1}\Delta\theta - \mathbf{v}\cdot\nabla\theta + \mathrm{Pe}\,F(\theta) = 0, \tag{2}$$

where $\mathbf{v}$ is the dimensionless velocity field and $\mathrm{Pe} = Uw/\kappa$ is the Peclet number, which is found to be large in the experiment (see Fig. 3). A standard choice for F is $F(\theta) \sim \theta(1-\theta^2)$, which leads to kink solutions for θ [11]; we restrict ourselves to this choice, which turns out to be noncrucial for our conclusions [12]. In the phase field model, F is related to the free energy of the system; in our case, it models the effect of the recirculating flow, inside the finger, which sustains the boundary layer. The use of (2) rather than the complete advection diffusion equation is the main point of this model. It allows us to impose the structure of the thermal field, without solving the full equations of the problem within the boundary layer and inside the plume. Concerning the boundary conditions, they are $\Psi = \pm\frac{1}{2}$ along the walls and $\theta = \pm 1$ far from the interface. At this stage, there is no sharp interface between the two fluids.

We now expand both fields θ and ψ inside and outside the boundary layer, in terms of the small parameter Pe^{-1}. At zeroth order, the boundary layer thickness is zero, so that it reduces to a line of discontinuity Γ which is a streamline and an isotherm. In the fluid, θ is either 1 or -1 so that one obtains the following equations for Ψ:

$$\Delta\Psi = 0. \tag{3}$$

Therefore, one recovers exactly the problem of a buoyancy-driven finger, with zero surface tension. The solution to this problem was found by Saffman and Taylor [1], in the form of a continuous set of solutions parametrized by λ. At this stage, one thus obtains the shape of the finger, but, as expected, one must go one order beyond to find the selection mechanism. At the same order, but within the boundary layer, one gets from Eq. (2) the following equation for θ:

$$\theta_{rr} + F(\theta) = 0, \tag{4}$$

where r is the stretched coordinate normal to Γ, using Pe^{-1} as the characteristic length. At the next order, one gets a linear nonhomogeneous differential equation of second order, involving θ_r and ψ. The linear operator is self-adjoint, and one can write a solvability condition [11] which ensures the convergence of the expansion. By applying this constraint, and matching the inner and outer expansions, one finally gets the boundary conditions for the flow outside the finger:

$$\frac{\partial\phi}{\partial n} = -\frac{1}{\mathrm{Pe}}\frac{\partial\tau}{\partial s}(1 + \alpha C\cos\tau), \tag{5}$$

where τ is the angle between the normal at the interface and the vertical y axis, n is the coordinate normal to the interface, s is the arclength, ϕ is the velocity potential, and C is a positive constant of order 1 which depends on the particular form of F. Concerning the velocity potential ϕ, one has continuity across the interface. The right-hand side of Eq. (5), which is proportional to the local curvature $\partial\tau/\partial s$, represents the effect of the thermal diffusivity. It expresses the fact that fluid flow trajectories cross the interface, in a way somewhat similar to the viscous fingering with a film drained by the interface [13]. In our case, this comes from the fact that isotherms are not streamlines when the thermal diffusion is taken into account. The present problem is therefore different from ordinary Saffman-Taylor; however, we find that the perturbation, represented in the right-hand side of Eq. (5), also induces a selection. This has been checked both numerically and analytically [14]; we present herein only the results obtained analytically, leading to the following estimates: When the quantity αC is smaller than 1, one gets

$$\lambda - 0.5 \approx \alpha^{-1}\mathrm{Pe}^{-1},$$

whereas in the opposite case we have

$$\lambda - 0.5 \approx C^{2/3}\mathrm{Pe}^{-2/3}.$$

We thus obtain fingers close to one-half, which is in good agreement with the experiment.

To summarize, we have revealed the existence of

Saffman-Taylor plumes, i.e., thermal plumes which have the same form, size, and dynamics as Saffman-Taylor fingers, and shown that the small parameter which leads to shape selection is the thermal diffusivity.

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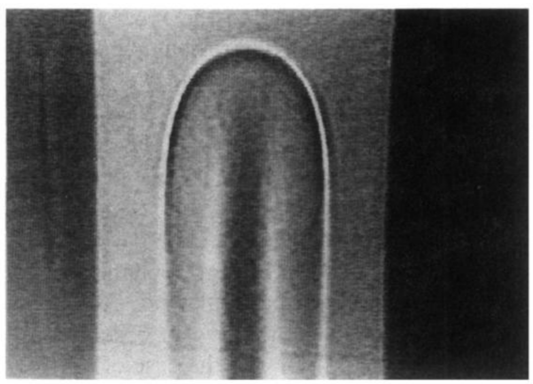

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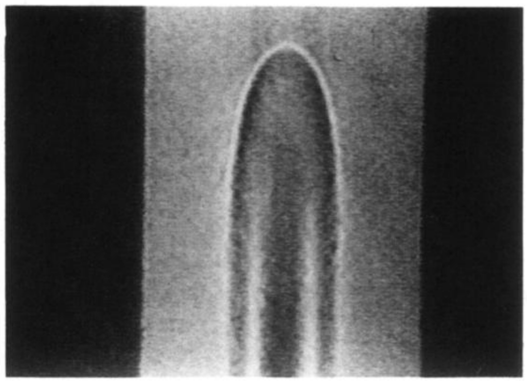
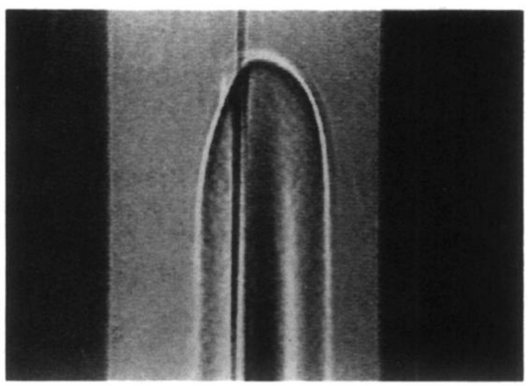
(a)



(b)



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