Nonlinear Theory of Localized Standing Waves

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(Received 20 May 1992)

An investigation of the nonlinear dispersive equations of continuum mechanics reveals localized standing-wave solutions that are domain walls between regions of different wave number. These states can appear even when the dispersion law is a single-valued function of the wave number. In addition, we calculate solutions for kinks in cutoff and noncutoff modes, as well as cutoff breather solitons.

PACS numbers: 43.25.+y, 03.40.Kf

Experiments on continuous media have demonstrated the existence of propagating envelope solitons [1,2], Scott Russell–type solitons [3], as well as standing solitons such as breathers [4] and kinks in cutoff modes [5]. These localized modes are described by equations that, in leading order, are closely allied to the nonlinear Schrödinger (NLS) and Korteweg–de Vries (KdV) equations [6]. The recent observation of domain walls [7] has motivated us to investigate the general theory of modulated standing waves in a continuous medium. We are especially interested in the fact that the domain wall breaks parity as well as translational invariance. We report below the finding that the domain wall and noncutoff kinks are described by a new set of modulational equations (i.e., not the NLS, KdV, Toda, or sine-Gordon equations). Our calculations are limited to nontopological soliton, localized states.

We consider the general class of nonlinear dispersive continuum mechanical systems that can be cast in the form

$$\rho'_{tt}+\hat{\omega}_0^2\rho'=\hat{G}\rho'^3\,, \tag{1}$$

where $\hat{\omega}_0^2$ and $\hat{G}$ are linear isotropic differential operators in $\mathbf{\nabla}$, i.e., are functions of ∇^2; the subscript t denotes differentiation with respect to time. The equations of motion for acoustic and optical phonons, the flexing modes of plates and shells, the continuum limit of the pendulum lattice, as well as many other systems can, to a reasonable approximation, be written in form (1). Our investigations will be limited to one space dimension plus time. Even with this restriction, the results may have applications to information storage and to fiber optic communication [8], and hopefully may also point the way to generalizations in higher dimensions.

We seek solutions to (1) in a form that allows for the modulations of both amplitude and wave number $k=\theta_x$:

$$\rho'(x,t)=\psi(x,t)\cos\theta(x,t)e^{-i\omega t}+\text{c.c.}\,, \tag{2}$$

where ω is a constant (the frequency of the standing wave being modulated), ψ and θ are complex, and c.c. denotes the complex conjugate of the preceding expression.

To derive approximate modulational equations of motions for θ and ψ, we assume that $\psi=O(\epsilon)$, where $0<\epsilon\ll 1$, and assume that time derivatives are small compared with space derivatives. By substituting (2) into (1) and equating the coefficients of $\sin\theta$ and $\cos\theta$ separately to zero (this is permissible when $k\sim 1$), we obtain

$$2i\omega\psi_t=[\omega_0^2(k)-\omega^2-\tfrac{9}{4}G|\psi|^2]\psi-\tfrac{1}{2}g_k\psi_{xx}-g_{kk}[\tfrac{1}{2}k_x\psi_x+\tfrac{1}{6}k_{xx}\psi]-\tfrac{1}{8}g_{kkk}k_x^2\psi\,, \tag{3}$$

$$2i\omega\psi\theta_t+g^{1/2}(g^{1/2}\psi)_x=0\,, \tag{4}$$

where the subscript x stands for differentiation with respect to x, $\omega_0(k)$ is the dispersion relationship governing infinitesimal waves of constant k, and $g(k)=d\omega_0^2/dk$. We shall assume that ω_0 is a monotonically increasing function of k, so that $g\geq 0$. There is no significant loss of generality here, for the issue we face is that of understanding how two modes of different wave number can be in equilibrium, even when ω_0 is a single-valued function of k. We suppose that $\hat{G}$ is a constant (G); progress can also be made, at a cost of greater complexity, for more general $\hat{G}$.

Equations (3) and (4) provide a description of how smooth modulations of standing waves develop in space and time. For a stationary state they imply

$$\psi\equiv\psi_1[g_1/g]^{1/2}\,, \tag{5}$$

$$a+bk_{xx}+\tfrac{3}{4}\frac{db}{dk}k_x^2=0\,, \tag{6}$$

where ψ_1 and g_1 are real constants and

$$a(k)=\frac{3\{[\omega^2-\omega_0^2(k)]g(k)+9Gg_1|\psi_1|^2/4\}}{[g(k)]^2}\,, \tag{7}$$

$$b(k)=-g^{1/2}\frac{d^2}{dk^2}g^{-1/2}. \tag{8}$$

Consider now solutions in which k asymptotes to $k_{+\infty}$ for $x\to+\infty$, and in which $k\to k_{-\infty}$ for $x\to-\infty$. From (6), we see that $a(k_{+\infty})=a(k_{-\infty})=0$. For the kink $k_{+\infty}=k_{-\infty}$, but for the domain wall $k_{+\infty}\neq k_{-\infty}$ [so that in this case $a(k)=0$ must have multiple solutions]. The restriction $a(k_{\pm\infty})=0$ is a unification of two physical conditions that are (1) at finite amplitude, the resonant frequency shifts to ω, where $\omega^2=\omega_0^2(k)-\tfrac{9}{4}G|\psi|^2$; and

(2) the amplitude and wave number must adjust so that the flux of the adiabatic invariant $(U/\omega)d\omega/dk$, where U is the energy density, vanishes in the steady state (5).

In the case of the domain wall, a direct integration of (6) yields the key connection relation

$$\int_{k_{-\infty}}^{k_{+\infty}} a(k)b^{1/2}(k)dk=0\,. \tag{9}$$

A simple interpretation of (9) can be obtained by the transformation $(k,x)\rightarrow(x,t)$ which converts the solution to (6) into the motion of a zero-energy particle of unit mass in the potential

$$V(k,k_-)=\frac{1}{b^{3/2}(k)}\left\{\int_{k_-}^{k} a(\kappa)b^{1/2}(\kappa)d\kappa - \tfrac{1}{2}b^{3/2}(k_-)k_x^2(k_-)\right\}, \tag{10}$$

where k_- is a reference value of k. Should this potential have a maximum at some value of k (say, k_-), then the constant of integration can be chosen so that $k_x(k_-)=0$. Since $\partial k_x^2/\partial k=2k_{xx}$, it follows that $k_{xx}(k_-)=a(k_-)=0$, so that $k\rightarrow k_-$ as $x\rightarrow -\infty$. When the potential has a second maximum (at k_+, say), then a particle starting from rest at k_- in the potential (10) will just reach k_+ if $V(k_+,k_-)=V(k_-,k_-)$, i.e., if (9) holds. This corresponds to a domain wall with $k_+=k_{+\infty}$. If $V(k_+,k_-)>V(k_-,k_-)$, the particle returns to k_-. This corresponds to a kink. In order for V to have a maximum, at $k=k_-$, say, it is necessary that $g^2+\frac{9}{4}G|\psi|^2 g_k>0$. The above interpretation applies for $b>0$; a similar analysis can be carried out if $b<0$ but, if b passes through zero, this simple picture breaks down.

The domain-wall profile can be calculated analytically when the net change in k and ω^2 are both small, i.e., when $k'\equiv k-k_1=O(\delta)\ll 1$ and $g=O(\epsilon^2)$, so that

$$\omega_0^2=\omega_1^2[1+\epsilon^2(\gamma_1 k'+\gamma_2 k'^2+\gamma_3 k'^3+\gamma_3 k'^4+\cdots)]\,, \tag{11}$$

where k_1 and $\omega_1=\omega_0(k_1)$ are constants. In this case, a domain-wall solution exists in the form

$$k=k_1+\Delta k'\tanh[D\Delta k'(x-x_0)]\,, \tag{12}$$

where $D=5-\gamma_1^2\gamma_4/\gamma_2^3$, provided that the dispersion satisfies the constraint $2\gamma_2^2=\gamma_1\gamma_3$, or equivalently

$$\left[\frac{d^2}{dk^2}g^{-2}\right]_{k=k_1}=0\,. \tag{13}$$

Substitution of (11) into (5) and (2) yields the complete solution

$$\rho'=\frac{2}{3}\left(\frac{2\omega_1(\omega_1-\omega)}{G}\right)^{1/2}\left\{1-\frac{\gamma_2}{\gamma_1}\Delta k'\tanh[D\Delta k'(x-x_0)]\right\}\sin\left\{k_1(x-x_0)+\frac{1}{D}\ln\cosh[D\Delta k'(x-x_0)]\right\}. \tag{14}$$

Figure 1 shows an example of a domain-wall solution given by this theory.

The domain-wall solution to the nonlinear standing-wave equations has been obtained by requiring that the expansion of a/b as a power series in k' be $O(\delta^3)$. This restriction is imposed by the connection relation (9) and can only be met when (13) is satisfied along with

$$\omega-\omega_1=\tfrac{1}{2}\epsilon^2\left\{\frac{\gamma_1^2\omega_1}{2\gamma_2}+\gamma_2\omega_1 D\Delta k'^2\right\},$$
$$|\psi_1|=\frac{2\sqrt{2}}{3}\left(\frac{\omega_1(\omega_1-\omega)}{G}\right)^{1/2}. \tag{15}$$

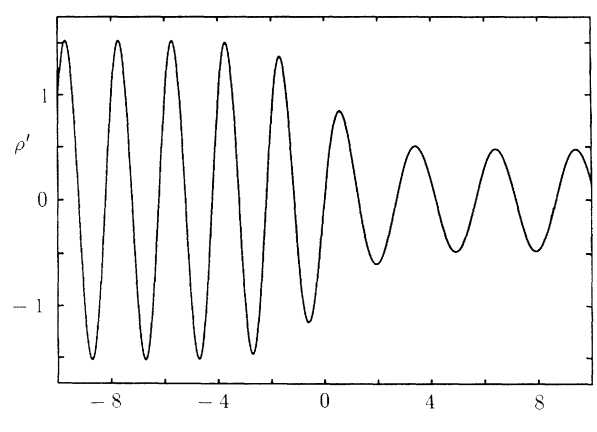

FIG. 1. Snapshot of a standing domain wall between modes of wavelength 2 for $x\rightarrow -\infty$ and waves of wavelength 3 for $x\rightarrow\infty$. This is a plot of ρ' given by (2), (5), and (11) for $x_0=0$, $\gamma_1=\gamma_2$, $D=1$, $\Delta k'=\pi/6$, and $k_1=5\pi/6$. The ordinate is ρ' in units of $\frac{2}{3}(-2/G\gamma_2)^{1/2}\epsilon\omega_1\gamma_1$.

The strong restriction (13) is not applicable to the kinks. In this case one need only require that $a/b=O(\delta^2)$. Setting now

$$\omega-\omega_1=\tfrac{1}{2}\epsilon^2\left\{\frac{\gamma_1^2\omega_1}{2\gamma_2}+\gamma_1\left[1-\frac{\gamma_1\gamma_3}{2\gamma_2^2}\right]\omega_1\bar{k}'\right\}, \tag{16}$$

we obtain the stationary solution

$$\theta=\theta(x_0)+k_1(x-x_0)+\left(\frac{\bar{k}'}{K}\right)^{1/2}\tanh[(K\bar{k}')^{1/2}(x-x_0)]\,, \tag{17}$$

where

$$K=\frac{\gamma_1(\gamma_1\gamma_3-2\gamma_2^2)}{4\gamma_2(\gamma_1\gamma_3-\gamma_2^2)}\,. \tag{18}$$

The spatial phase shift between $x=-\infty$ and $x=+\infty$ is $2(\bar{k}'/K)^{1/2}$. When changes in k are $O(1)$, it may be necessary to restore the term $-\theta_t^2\psi$ on the right-hand side of (3).

In the two solutions just obtained, $\Delta k'$ and $\bar{k}'$ are free parameters that characterize the discontinuity of the domain wall and phase shift of the kink. Also, the scaling of ψ (i.e., ϵ) is independent of δ, which scales the spatial variations. Figure 2 shows a kink soliton given by this theory.

The algebraic details of the solutions for a kink and for a domain wall are given elsewhere [9], where time-dependent solutions are also considered.

Turning now to possible physical realizations of noncutoff kinks, we first consider the pendulum lattice [7] and optical phonons for which the dispersion law is of the form $\omega_0^2=\omega_{0,0}^2+(\omega_D^2-\omega_{0,0}^2)\sin^2(ka/2)$, where $\omega_D \equiv\omega_0(ka=\pi)$. According to (8), $b<0$ for this dispersion law, and a necessary condition for a kink solution is $g^2+\frac{9}{4}G|\psi|^2g_k<0$. This clearly excludes low-amplitude kink solutions when g is $O(1)$, and these systems cannot possess standing noncutoff kinks described by (3) and (4) unless $|\omega_D^2-\omega_{0,0}^2|/\omega_{0,0}^2\ll 1$. Similarly, $b<0$ for flexing modes, but the dispersion law for flexing shells [1] has a minimum at finite k. Near this wave number, g is small, and again such a system may have localized states.

In the presence of damping and parametric drive, (1) becomes

$$\frac{\partial^2\rho'}{\partial t^2}+\hat{\omega}_0^2\rho'+2\gamma\rho'\cos 2\omega t+\beta\frac{\partial\rho'}{\partial t}=\hat{G}\rho'^3. \tag{19}$$

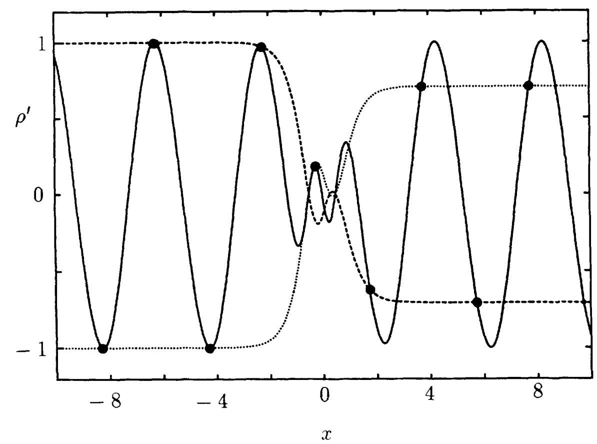

FIG. 2. Snapshot of a kink in a mode of wavelength 4. This is a plot of ρ' as given by (2), (5), and (15), for $x_0=0$, $\gamma_1=\gamma_2$, $K=1$, and $\bar{k}'=9\pi^2/64$. To make the kink profiles more apparent, two sets of points separated by half-wavelength intervals are indicated, and the curves of constant phase joining these points are shown dashed.

In the steady state (with θ real), the right-hand side of (3) is supplemented by the additional term $\gamma\psi^*-i\beta\omega\psi$. The steady state is now characterized by a phase lag, S, relative to the drive, where

$$\sin 2S=-\omega\beta/\gamma, \tag{20}$$

and $a(k)$ in (7) must be replaced by

$$\hat{a}(k)=\frac{3[\{\omega^2-\omega_0^2(k)+(\gamma^2+\beta^2\omega^2)^{1/2}\}g(k)+9Gg_1|\psi_1|^2/4]}{[g(k)]^2}. \tag{21}$$

When the dispersion law possesses a finite cutoff at long wavelengths, a solution of the form (2) exists for $\theta=0$, so that only the amplitude is modulated. In this case (1) gives

$$2i\omega\psi_t=[\omega_0^2(0)-\omega^2-3G|\psi|^2]\psi-c^2\psi_{xx}, \tag{22}$$

where we have set

$$\omega_0^2=\omega_0^2(0)+c^2k^2+\cdots. \tag{23}$$

Equation (22) is a well-known NLS equation. It possesses a nonpropagating breather soliton for $G>0$,

$$\psi=\left[\frac{2[\omega_0^2(0)-\omega^2]}{3G}\right]^{1/2}\operatorname{sech}\left\{\left[\frac{\omega_0^2(0)-\omega^2}{c^2}\right]^{1/2}(x-x_0)\right\}, \tag{24}$$

and a nonpropagating lower cutoff kink for $G<0$,

$$\psi=\left[\frac{\omega^2-\omega_0^2(0)}{-3G}\right]^{1/2}\tanh\left\{\left[\frac{\omega^2-\omega_0^2(0)}{2c^2}\right]^{1/2}(x-x_0)\right\}. \tag{25}$$

Our analysis of the modulational equations for standing waves leads us to the conclusion that such systems can display a wealth of localized states. The solutions presented here have the advantage of being analytic together with the disadvantage of not being the most general solutions to (3) and (4). Indeed, the observations of domain walls which prompted this research dealt with a system for which (13) does not apply. We conclude that the parameter space for these new localized states must be richer than has emerged from our leading-order solutions to (3) and (4).

One of us (S.J.P.) thanks the Division of Engineering and Geophysics of the Office of Basic Energy Science of

U.S. DOE for support. We wish to acknowledge a valuable discussion with Joseph Rudnick. Two of us (B.D. and A.L.) were supported from the NPS Direct-Funded Research Program. We are grateful to Maureen Roberts for help in preparing this manuscript.

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