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Solitons, Euler's Equation, and Vortex Patch Dynamics

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Integrable systems related to the Korteweg–de Vries (KdV) equation are shown to be associated with the dynamics of vortex patches in ideal two-dimensional fluids. This connection is based on a truncation of the exact contour dynamics analogous to the "localized induction approximation" which relates the nonlinear Schrödinger equation to the motion of a vortex filament. Single-soliton solutions of the periodic modified KdV problem correspond to uniformly rotating shapes which approximate the Kirchoff ellipse and known generalizations. A simple geometrical interpretation of the dual Poisson bracket structure of the modified KdV hierarchies is given.

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One of the more remarkable results in the mathematical physics of Hamiltonian systems is that the nonlinear Schrödinger (NLS) equation,

$$i\psi_t = -\psi_{ss} - \tfrac{1}{2}|\psi|^2\psi, \tag{1}$$

describes the motion of a nonstretching vortex filament moving in three dimensions under a local approximation to Euler's equation of inviscid hydrodynamics [1]. Here, the complex quantity $\psi(s,t)$ is given by the curvature $\kappa(s,t)$ and torsion $\tau(s,t)$ of the curve at arclength position s and time t as

$$\psi(s,t) = \kappa(s,t)\exp\left[i\int^s ds'\,\tau(s',t)\right]. \tag{2}$$

Within this local approximation, (1) embodies the motion of a point $\mathbf{r}(s,t)$ on the curve in the form

$$\mathbf{r}_t = U\hat{\mathbf{n}} + V\hat{\mathbf{b}} + W\hat{\mathbf{t}}, \tag{3}$$

where the tangent, normal, and binormal vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$ comprise the Frenet-Serret triad and the velocity functions in the local approximation are $V=\kappa$ and $U=W=0$. Unlike Euler's equation, the NLS is known to be an integrable system with an infinite number of conserved quantities [2].

The existence of other, but more formal, connections between integrable systems and Eulerian hydrodynamics has been known for some time in the context of the Korteweg–de Vries (KdV) equation, whose Poisson bracket structure [3] is similar to that of Euler's equation [4,5]. Yet, the relation between KdV dynamics and the actual motion of an inviscid, incompressible fluid has remained unclear. In this regard, however, a recent study [6] has noted the equivalence of the modified Korteweg–de Vries (mKdV) hierarchy and a certain class of local dynamics of closed curves in the plane and pointed out two intriguing connections with *two-dimensional* Eulerian flows: (i) conservation of enclosed area, consistent with incompressibility, and (ii) conservation of circulation, consistent with the Kelvin circulation theorem.

Here we show that the connection between KdV dynamics and Euler flows in two dimensions parallels that involving the NLS in three dimensions; i.e., it is through a local approximation. The elementary distribution of vorticity of interest in two dimensions is a vortex *patch* [7], a bounded region of constant vorticity surrounded by irrotational fluid. Starting from the exact nonlocal evolution equation for the boundary of such a domain [8] we find that within a local approximation the boundary evolution coincides with that which was earlier shown [6] to be equivalent to the KdV dynamics. The curve motion has a general form like the space curve described in Eq. (3), but with motion only in the normal and tangential directions,

$$\mathbf{r}_t = U\hat{\mathbf{n}} + W\hat{\mathbf{t}}, \tag{4}$$

where now $U=\kappa_s$ and $W=-\tfrac{1}{2}\kappa^2$. The curvature then evolves according to the modified KdV equation,

$$\kappa_t = -\kappa_{sss} - \tfrac{3}{2}\kappa^2\kappa_s, \tag{5}$$

which is integrable.

An alternative derivation of these results begins with the Hamiltonian formulation of ideal hydrodynamics. For a vortex patch, the Hamiltonian $\mathcal{H}$ can be expressed as a functional of the patch boundary $\mathbf{r}(s)$, with a long-range coupling between tangent vectors. This allows the dynamics to be written in a simple variational form

$$\hat{\mathbf{n}}\cdot\mathbf{r}_t=\partial_s\left(\hat{\mathbf{n}}\cdot\frac{\delta\mathcal{H}}{\delta\mathbf{r}}\right). \tag{6}$$

A local approximation like that described above leads to an expansion of the Hamiltonian in powers of the curvature and its derivatives in the form

$$\mathcal{H}=\oint ds\,(a+b\kappa^2+c\kappa^4+d\kappa_s^2+\cdots). \tag{7}$$

The coefficients a, b, etc., depend on some (arbitrary) large-scale cutoff in the truncation scheme. These leading-order terms are in fact equivalent to the Hamiltonians of the first few members of the mKdV hierarchy. Moreover, the existence of a dual Poisson bracket structure [9] to the mKdV dynamics follows from the possibility of viewing these and the successive mKdV Hamiltonians as functionals of $\mathbf{r}(s)$, as in Eq. (6) above, or of $\kappa(s)$. Note that the first two energies in the expansion (7) describe a contour endowed with surface tension [10] and elasticity [11,12]. The *single-soliton* solutions of the variational dynamics (6) are extrema of these energies subject to constraints of fixed length and/or area [13]. In the vortical fluid interpretation, such solutions are analogs of uniformly rotating vortex patches, the Kirchoff ellipse, and the "V states" of Deem and Zabusky [14].

Finally, we show that when the dynamics are recast in terms of the position of the boundary in the complex plane, $z(s,t)=x(s,t)+iy(s,t)$, a number of conservation laws and invariance properties arise quite naturally as counterparts of those of exact two-dimensional Eulerian dynamics. In particular, the appearance of the Schwarzian derivative [15,16] of $z(s,t)$ in the dynamics leads directly to the $SL(2,C)$ invariance of the motion.

Contour dynamics for vortex patches.— As first derived by Zabusky and coworkers [8], a point on the boundary of a vortex patch with vorticity ω_p moves with velocity

$$\mathbf{r}_t(s,t)=\frac{\omega_p}{2\pi}\oint ds'\ln\left(\frac{|\mathbf{r}(s,t)-\mathbf{r}(s',t)|}{r_0}\right)\hat{\mathbf{t}}(s',t). \tag{8}$$

The arbitrary parameter r_0 does not affect the dynamics because the integral of the tangent vector is zero for a closed curve. In the localized induction approximation we expand the terms in the integrand up to quadratic order in powers of $\Delta=s'-s$, and use the Frenet-Serret equations, $\hat{\mathbf{t}}_s=-\kappa\hat{\mathbf{n}}$, and $\hat{\mathbf{n}}_s=\kappa\hat{\mathbf{t}}$, to obtain

$$\mathbf{r}(s')\simeq\mathbf{r}(s)+\Delta\hat{\mathbf{t}}(s)-\tfrac{1}{2}\Delta^2\kappa\hat{\mathbf{n}}(s)+\cdots,$$

and likewise for $\mathbf{t}(s',t)$. The integral over s' in (8) is then truncated with a cutoff Λ at $s'=s\pm\Lambda/2$, allowing the integration to be done term by term. The dynamics may then be written in the form (4) where the normal and tangential velocities are

$$U=-A_1\kappa-\tfrac{1}{2}A_2\kappa_s+\cdots$$

and

$$W=A_0-\tfrac{1}{2}(A_2-\tfrac{1}{2}B_2)\kappa^2+\cdots,$$

where the coefficients $A_n(\Lambda)$ and $B_n(\Lambda)$ are elementary integrals [17].

The coefficients A_n and B_n for n odd are zero, by symmetry; those for n even do not require the introduction of a small-scale cutoff to remain finite, in contrast to the vortex line problem [18]. The parameter r_0 only affects the time scale of the motion, but it is convenient to choose it such that the condition $W_s=-\kappa U$ is satisfied, as this is the *gauge* in which the arclength parametrization is time independent. A Galilean transformation removes the term A_0 in W. The modified KdV dynamics in (5) follows from the general relation [6] between curvature evolution and the normal velocity in this gauge,

$$\kappa_t=-(\partial_{ss}+\kappa^2+\kappa_s\partial^{-1}\kappa)U\equiv-\Omega U. \tag{9}$$

Hamiltonian formulation.— In inviscid fluid dynamics, the Hamiltonian $\mathcal{H}$ is simply the kinetic energy; in two dimensions it may be expressed as a functional of the vorticity $\omega(\mathbf{r})$ as

$$\mathcal{H}[\omega]=\tfrac{1}{2}\int d^2r\int d^2r'\,\omega(\mathbf{r})\omega(\mathbf{r}')\mathcal{G}(\mathbf{r},\mathbf{r}'). \tag{10}$$

Here, $\mathcal{G}(\mathbf{r},\mathbf{r}')=\ln(|\mathbf{r}-\mathbf{r}'|/r_0)$ is the Green's function for the Poisson equation $\nabla^2\psi=-\omega$ obeyed by the stream function ψ. The time evolution of the vorticity, or indeed any function $f(\omega)$ follows from the Hamiltonian as $f_t=\{f,\mathcal{H}\}$ where the Poisson bracket of two functions of the vorticity is defined by [4,5]

$$\{F,G\}=\int d^2r'\,\omega(\mathbf{r}')\left(\nabla'\frac{\delta F}{\delta\omega}\right)\times\left(\nabla'\frac{\delta G}{\delta\omega}\right). \tag{11}$$

The vorticity evolution $\omega_t=-\mathbf{v}\cdot\nabla\omega$ follows from this and the relation $\psi(\mathbf{r})=\delta\mathcal{H}/\delta\omega$.

We now observe that for a region of constant vorticity, the Hamiltonian can be reexpressed as a functional of the boundary shape by repeated application of the divergence theorem,

$$\mathcal{H}_p[\mathbf{r}]=-\tfrac{1}{8}\omega_p^2\oint ds\oint ds'\,\hat{\mathbf{t}}(s)\cdot\hat{\mathbf{t}}(s')\Phi(\mathbf{R}), \tag{12}$$

where $\mathbf{R}=\mathbf{r}(s)-\mathbf{r}(s')$ and $\Phi(\xi)=\xi^2\ln(|\xi|/er_0)$. Expanding the integrand for s' near s as in the local approximation to the contour dynamics, and performing those integrals in the same manner leads to the curvature expansion for $\mathcal{H}$ given in Eq. (7).

For the case of a vortex patch, it is not necessary to consider *arbitrary* functional variations of the vorticity, but only those which preserve the piecewise constancy of ω. These, in turn, are associated with motion of the boundary normal to itself, suggesting the identification

$$\frac{\delta}{\delta\omega(\mathbf{r})} \rightarrow \frac{1}{\omega_p}\hat{\mathbf{n}}(s)\cdot\frac{\delta}{\delta\mathbf{r}(s)}\,. \tag{13}$$

One may verify that this functional derivative applied to the boundary Hamiltonian (12) generates the correct stream function for the patch,

$$\psi(\mathbf{r}) = \omega_p \oint ds' [\mathbf{r}-\mathbf{r}(s')]\times\hat{\mathbf{t}}(s')\ln[|\mathbf{r}-\mathbf{r}(s')|/er_0]\,. \tag{14}$$

Returning to the Poisson bracket (11) and integrating by parts, we see the step function discontinuity in ω at the boundary renders $\nabla\omega(\mathbf{r}') = -\omega_p \oint ds' \delta(\mathbf{r}'-\mathbf{r}(s'))\hat{\mathbf{n}}(s')$. Only the tangential component of $\nabla(\delta G/\delta\omega)$ then contributes, resulting in a Poisson bracket for two functionals of the boundary

$$\{F,G\} \equiv \oint ds' \left(\hat{\mathbf{n}}(s')\cdot\frac{\delta F}{\delta\mathbf{r}(s')}\right)\partial_{s'}\left(\hat{\mathbf{n}}(s')\cdot\frac{\delta G}{\delta\mathbf{r}(s')}\right). \tag{15}$$

This is indeed very similar to the KdV Poisson bracket [3-5]. If we set $F=\mathbf{r}(s,t)$ and $G=\mathcal{H}$, the we recover the boundary dynamics in Eq. (6) [17].

The dynamics generated by successive terms in the curvature expansion of $\mathcal{H}$ in (7) leads to the first few members of the mKdV hierarchy. That is, $\mathcal{H}=a\oint ds$ gives $U=a\kappa_s$, associated with the ordinary mKdV equation. The elasticity Hamiltonian $\mathcal{H}=b\oint ds\,\kappa^2$ leads to $U=2b(\kappa_{sss}+\frac{3}{2}\kappa^2\kappa_s)$, which gives the third member of the mKdV hierarchy [6]. By suitably tuning the coefficients c and d in (7) (by choice of r_0), the next member is also found.

While each of these Hamiltonians has been viewed as a functional of the contour $\mathbf{r}(s)$, the vector $\mathbf{r}(s)$ itself does not appear explicitly, so they may be considered functionals of κ alone. Indeed, it is known [9] that the mKdV equations may be written as

$$\kappa_t = \partial_s\left(\frac{\delta\mathcal{T}^{(n)}}{\partial\kappa}\right), \tag{16}$$

where the $\mathcal{T}^{(n)}$ are the successive Hamiltonians of the hierarchy. It is then natural to ask about the relation between the two forms of curve dynamics (6) and (16). This relation is provided by a fundamental geometric identity connecting functional derivatives with respect to $\mathbf{r}(s)$ and with respect to $\kappa(s)$ [11,17],

$$\partial_s\left(\hat{\mathbf{n}}(s)\cdot\frac{\delta}{\delta\mathbf{r}(s)}\right) = \Omega\,\partial_s\left(\frac{\delta}{\delta\kappa(s)}\right), \tag{17}$$

where the operator Ω is defined in (9). Thus, with $\mathcal{H}^{(1)}=\frac{1}{2}\oint ds\,\kappa^2$ we obtain the mKdV dynamics from

$$\kappa_t = \partial_s\left(\hat{\mathbf{n}}\cdot\frac{\delta\mathcal{H}^{(1)}}{\delta\mathbf{r}}\right), \tag{18}$$

whereas with $\mathcal{H}^{(2)} = -\oint ds(\kappa^4/8-\kappa_s^2/2)$ the same dynamics arises from

$$\kappa_t = \partial_s\left(\frac{\delta\mathcal{H}^{(2)}}{\delta\kappa}\right). \tag{19}$$

Together, Eqs. (18) and (19) imply the existence of two distinct Poisson brackets for the hierarchy. They are

$$\{\kappa(s),\kappa(s')\}_1 = \partial_{s'}\delta(s-s') \tag{20}$$

and

$$\{\kappa(s),\kappa(s')\}_2 = [\partial_{ss}+\kappa^2(s)]\partial_{s'}\delta(s-s') + \kappa_s(s)\int^s ds''\kappa(s'')\partial_{s''}\delta(s''-s')\,. \tag{21}$$

Motion in the complex plane.— A corollary to the interpretation of the mKdV hierarchy as *curvature* dynamics is the geometric significance of the KdV hierarchy, which is found by considering curve dynamics in the complex plane. Recognizing that the unit tangent and normal are, respectively, z_s and $-iz_s$, the complex version of the general equation of motion (4) is $z_t=(W-iU)z_s$. The nonlinear transformation discovered by Miura [19] allows one to form a correspondence between solutions of the mKdV equation and of the KdV equation. If $\kappa(s,t)$ satisfies the mKdV equation, then $u=-\frac{1}{2}\kappa^2-i\kappa_s$ satisfies the KdV equation $u_t=-u_{sss}+3uu_s$. Using the complex representation of the curvature $\kappa=-i(z_{ss}/z_s)$, we find that

$$u = -\left[\left(\frac{z_{ss}}{z_s}\right)_s - \frac{1}{2}\left(\frac{z_{ss}}{z_s}\right)^2\right] \equiv -\{z,s\}\,. \tag{22}$$

We recognize [15] the quantity $\{f,x\}$ as the Schwarzian derivative of f with respect to its argument x.

It follows that the dynamics (4) and (5) for mKdV may be written as

$$z_t = -\{z,s\}z_s = -z_{sss}+\tfrac{3}{2}z_s^* z_{ss}^2\,. \tag{23}$$

In this form, it is possible to see that the localized induction approximation preserves some of the basic conserved quantities of the exact vortex patch dynamics [18]. Among these are the following, expressed both as area integrals over the patch $\mathcal{P}$ and contour integrals: (i) area,

$$\int_{\mathcal{P}} d^2r \propto \oint ds\,\frac{z}{z_s}\,; \tag{24}$$

(ii) components of the center of mass,

$$\int_{\mathcal{P}} d^2r\,\mathbf{r} \propto \oint ds\,\frac{z^2}{z_s}\,; \tag{25}$$

and (iii) angular momentum,

$$\int_{\mathcal{P}} d^2r(x^2+y^2) \propto \oint ds\,\frac{|z|^2 z}{z_s}\,. \tag{26}$$

The constancy of each follows from the dynamics in the form (23).

The Schwarzian derivative is invariant under fractional linear transformations in the complex plane [15]. That is, $\{z,s\}=\{w,s\}$ under

$$z \rightarrow w=(az+b)/(cz+d) .$$

Since z_t/z_s is also invariant, the mKdV dynamics $z_t = -\{z,s\}z_s$ retains its form and becomes $w_t=-\{w,s\}w_s$. However, in the w plane, the curve is no longer evolving by mKdV dynamics, since s is not its arclength. For instance, the quantity $\oint ds\, w/w_s$ is conserved, but it is not the area, as it would be if s were arclength for the curve in the w plane. Nonetheless, the dynamics is integrable, since it is equivalent to the z dynamics. The recursion relations [6,20] which connect the velocity functions $U^{(n)}$ and $W^{(n)}$ of the hierarchy allow one to prove by induction [17] that all the functions $W^{(n)}-iU^{(n)}$ are invariant under SL$(2,C)$ transformations.

Contours associated with periodic solutions of the mKdV equation.— Finally, let us turn to the meaning of the soliton solutions to the curvature evolution equations. Single-soliton solutions have the traveling wave form $\kappa(s,t)=g(z)$, where $z=s-ct$. If we now consider dynamics in the form of Eq. (18), and recall that the functional derivatives of the length L and area A are $\delta L/\delta\mathbf{r}=\kappa\hat{\mathbf{n}}$ and $\delta A/\delta\mathbf{r}=\hat{\mathbf{n}}$, we see such solutions satisfy

$$\hat{\mathbf{n}}\cdot\frac{\delta}{\delta\mathbf{r}}(\mathcal{H}+cL+dA)=0 , \tag{27}$$

for some constant d. This is the constrained extremization problem for $\mathcal{H}$ with Lagrange multipliers c and d conjugate to the length and area [11]. Note that this connection between traveling waves and constrained minimization holds for *any* $\mathcal{H}$.

For the particular case of the mKdV dynamics, the function g obeys an ordinary differential equation which may be integrated twice and rescaled in terms of the variable $h=g/2$ to yield

$$h_z^2=P(h)=-h^4+ch^2+ah+b , \tag{28}$$

where a and b are integration constants. Since the polynomial $P(h)$ does not have a cubic term, only three of its roots are independent; these determine the structure of the solutions. For a given closed curve of perimeter L, containing n periods of h, there is a one-parameter family of solutions specified by the area [11,17]. For small deviations from a circle, these shapes closely approximate the exact solutions (such as the Kirchoff ellipse for $n=2$), although they deviate when the maximum curvature is large.

In light of the present results, it is natural to reconsider the Hamiltonian formulation of three-dimensional fluid dynamics and its relation to that of the nonlinear Schrödinger equation. On a more general level, one might ask if the notion of a local expansion of chiral dynamics leading to integrability is relevant to other problems quite removed from hydrodynamics. The growth of "chiral crystals" [21] and the fluctuations of Fermi surfaces [22] are possible examples.

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