Random Transverse Field Ising Spin Chains

Daniel S. Fisher
Physics Department, Harvard University, Cambridge, Massachusetts 02138
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A renormalization-group analysis of the spin-$\frac{1}{2}$ transverse field Ising model with quenched randomness is presented; it becomes *exact* asymptotically near the zero temperature ferromagnetic phase transition. The spontaneous magnetization is found to vanish with an exponent $\beta=\frac{1}{2}(3-\sqrt{5})$, while in the disordered phase the typical and average spin correlations are found to decay with different correlation lengths, which diverge with exponents $\tilde{\nu}=1$ and $\nu=2$, respectively.

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Virtually the only exactly solvable statistical mechanical model with quenched randomness on a real lattice is the McCoy-Wu random Ising model [1,2] in which the random couplings only depend on *one* of the two coordinates (x). Although this appears extremely artificial as a 2D statistical mechanics model, in the continuum limit in the y direction, it is equivalent to the random transverse field quantum spin-$\frac{1}{2}$ Ising model with Hamiltonian

$$\mathcal{H}=-\sum_j J_j S_j^z S_{j+1}^z - \tfrac{1}{2}\sum_j h_j S_j^x \,, \tag{1}$$

where we have generalized to the case where the exchanges J_j, drawn independently from a distribution $\pi_0(J)dJ$, and the transverse fields h_j with distribution $\rho_0(h)dh$ are both random (but, without loss of generality, all positive). This is the simplest nontrivial interacting quantum model with quenched randomness. It undergoes a ferromagnetic phase transition at $T=0$ as the control parameter $\Delta\equiv\overline{\ln h}$ (where the overbar denotes averaging over the quenched variables) is decreased to Δ_c. One might hope that this transition would be prototypical of quantum phase transitions in random systems, but, as we shall see, it exhibits very unusual behavior.

The McCoy-Wu model is solved by transfer matrices in the *space* (x) direction [1,2]. Although the free energy and various other quantities are known [1,2], neither the spontaneous magnetization nor the spin correlations have been calculated. Nevertheless, rather strange behavior is found [1]: The susceptibility χ is infinite below a critical value Δ_χ which is bigger than Δ_c, due to the effects of rare regions which are anomalously strongly coupled. Thus the model appears to have (at zero temperature) three phases.

In this Letter, an approximate renormalization-group (RG) treatment of this model is given, which becomes exact asymptotically at long distances and low frequencies near to the critical point. Many new results are obtained, in particular the exponent β of the spontaneous magnetization $m\sim(\Delta_c-\Delta)^\beta$ and the behavior of both average and typical spin correlations which turn out to differ dramatically. Details of the calculations will be given in a longer paper [3]. From the exact solution, the principal correlations which are known are the surface spontaneous magnetization m_s of the end of a semi-infinite chain [$m_s\sim(\Delta_c-\Delta)^{\beta_s}$ with $\beta_s=1$] [1] and the long distance decay of the typical transverse correlations $C_{ij}^\perp\equiv\langle S_i^x S_j^x\rangle$ given by

$$\lim_{|i-j|\to\infty} -\ln C_{ij}^\perp/|i-j| \to \tilde{\xi}^{-1}$$

with probability 1, where $\tilde{\xi}\sim|\Delta-\Delta_c|^{-\tilde{\nu}}$ is the *typical correlation length* with $\tilde{\nu}=1$ (as for the pure case) [2]. The behavior of both m_s and $\tilde{\xi}$ supports the identification of $\Delta_c=\overline{\ln J}$ at the true transition point. At this point the system with $\pi_0=\rho_0$ is exactly self-dual. We shall see, however, that there is a *second* divergent length scale, ξ, associated with this transition.

To perform a renormalization-group transformation we follow a procedure close to that used by Dasgupta and Ma [4] for random Heisenberg antiferromagnetic chains: We first find the largest energy bonds and fields in the system with strength $\Omega=\Omega_0\equiv\max\{J_j,h_j\}$ and consider decimating away those with $\Omega-d\Omega\le J_j$ (or $h_j)\le\Omega$, thereby throwing out the high-energy information and focusing on the desired low-energy physics. If a given *strong bond* $J_j=\Omega$, we make the approximation that the two spins S_j and S_{j+1} are rigidly locked together as a *spin cluster* with an effective magnetic moment $\tilde{g}=g_j+g_{j+1}=2$ and an effective field $\tilde{h}\approx h_j h_{j+1}/\Omega<\Omega$ obtained from lowest-order perturbation theory in $h_{j(j+1)}/J_j$. For a *strong field* $h_j=\Omega$, on the other hand, we *eliminate* the *site* and get an effective bond strength $\tilde{J}\approx J_{j-1}J_j/\Omega<\Omega$ between the remaining now-nearest-neighbor sites $j-1$ and $j+1$. We then iterate this procedure with the maximum remaining energy scale Ω gradually being decreased. At each stage the remaining couplings $\tilde{J}$ and $\tilde{h}$ are *independent*, although $\tilde{h}$ and the magnetic moments $\tilde{g}$ are correlated. We must thus keep track of the distributions $\rho_\Omega(\tilde{h},\tilde{g})$ and $\pi_\Omega(\tilde{J})$. Because of the multiplicative form of the recursion relations, it is natural to work with logarithmic variables. Inspection of the resulting flow equations shows that the distributions get broader and broader even on a logarithmic scale. Thus the perturbative decimation approximation, which is initially not very good, becomes better and better under renormalization since, with high probability, the neighboring fields $\{\tilde{h}\}$ of a strong bond are much weaker than $\tilde{J}=\Omega$. Thus, if the width of the distributions of $\ln\tilde{J}$ and $\ln\tilde{h}$ grows without bound, the RG becomes *asymptotically exact*. As we shall see, this is indeed the case for Δ near Δ_c.

A simple change of variables *and a rescaling* brings the RG flow equations into a convenient form: We define $\Gamma\equiv-\ln(\Omega/\Omega_0)$, $\eta\equiv-\ln(\tilde{J}/\Omega)/\Gamma\geq 0$, $\theta\equiv-\ln(\tilde{h}/\Omega)/\Gamma\geq 0$, and $\mu\equiv g/\Gamma^{\phi}$, which we approximate as a continuous variable with ϕ a (positive) exponent to be determined. We then have for the distributions $P_\Gamma(\eta)$ and $B_\Gamma(\theta,\mu)$ the following flow equations:

$$\Gamma\frac{\partial P_\Gamma(\eta)}{\partial\Gamma}=P_\Gamma+(1+\eta)\frac{\partial P_\Gamma}{\partial\eta}+R_\Gamma(0)\int d\eta' P_\Gamma(\eta-\eta')P_\Gamma(\eta')+[P_\Gamma(0)-R_\Gamma(0)]P_\Gamma\,,$$
$$\Gamma\frac{\partial B_\Gamma(\theta,\mu)}{\partial\Gamma}=(1+\phi)B_\Gamma+(1+\theta)\frac{\partial B_\Gamma}{\partial\theta}+\phi\mu\frac{\partial B_\Gamma}{\partial\mu}+P_\Gamma(0)\int d\theta'\int d\mu' B_\Gamma(\theta-\theta',\mu-\mu')B_\Gamma(\theta',\mu')+[R_\Gamma(0)-P_\Gamma(0)]B_\Gamma\,, \tag{2}$$

where $R_\Gamma(\theta)\equiv\int d\mu\, B_\Gamma(\theta,\mu)$ is the distribution of θ.

We first concentrate on the critical point. By duality between h's and J's we expect that the critical fixed point will have $R^*=P^*$, so that, ignoring the effects of μ's, we need only find one function $P^*(\eta)$. This satisfies a simple nonlinear fixed point equation, which has a *family* of solutions corresponding to all possible choices of $P^*(0)$. Almost all of these, however, have power-law tails for large η. Analysis of the RG flows shows that these can only arise from highly singular initial distributions of the J_j's of the form $\pi_0(J)\sim J^{-1}(\ln J)^{-y}$ for small J. If we exclude such singular distributions (which will in any case have different critical behavior than the generic case) then the only attainable fixed point is remarkably simple: $P^*(\eta)=e^{-\eta}$ (with $\eta\geq 0$ by definition). Linearization about P^* shows that this distribution is stable in the self-dual subspace. The scaling of the width of the distribution of $\ln\tilde{J}$ as $\ln\Omega$ implies that, with probability that approaches 1 for small Ω, the perturbative decimation approximation becomes valid: Universal quantities obtained at and near the critical point should thus be *exact*. Errors made at early stages of the RG should only affect nonuniversal coefficients.

From $P^*(0)=R^*(0)=1$, we can find the relation between length scales and energy scales: The fraction of bonds remaining at scale $\Omega=\Omega_0 e^{-\Gamma}$ obeys $dn_\Gamma/d\Gamma=-[P_\Gamma(0)+R_\Gamma(0)]/\Gamma$ since $(d\Gamma/\Gamma)P_\Gamma(0)$ is the fraction of bonds decimated with a change $d\Gamma$. Asymptotically at Δ_c, $n_\Gamma\sim\Gamma^{-2P^*(0)}\sim\Gamma^{-2}$ yielding a length scale $L(\Omega)\sim(\ln\Omega)^2$ corresponding to a *dynamic critical exponent* $z=\infty$, somewhat analogous to "activated dynamic scaling" in random field magnets [5].

In order to find the spin correlations, we need to know the typical number of "active" spins, $g\sim\Gamma^{\phi}$, in the spin clusters at scale Γ. Each active spin—defined as those which were not decimated out before the scale at which the cluster was formed—will be strongly correlated with the other active spins in the cluster when the strong bonds which connect the cluster are decimated. Given $R^*(\theta)$ [and $P^*(0)$], we can adjust the exponent ϕ to find the appropriate full fixed-point distribution $B^*(\theta,\mu)$. Joint distributions cannot be found directly for arbitrary ϕ; however, examination of the conditional expectation of μ, $E(\mu|\theta)$, shows that the only allowed physically relevant fixed point corresponds to $\phi=(1+\sqrt{5})/2$, i.e., the golden mean. (Other ϕ's can only arise from initial distributions of g's with long tails, rather than the actual case for which all $g_j=1$ initially.) After some experimentation, a well-behaved solution with this ϕ can be found,

$$B^*(\theta,\mu)=\int\frac{d\nu}{2\pi}e^{-i\mu\nu}\exp[A(\nu)+\phi\theta\nu\, dA/d\nu-\theta]\,, \tag{3}$$

with $A(\nu)$ satisfying a simple differential equation which can be fully analyzed [3], yielding all the needed asymptotic behavior of B^*.

We are now in a position to find spin correlations $C_{ij}\equiv\langle S_i^z S_j^z\rangle$ at the critical point. Because the fraction of spins which has not been decimated at a length scale $L\sim\Gamma_L^{-2}$ is only of order $L^{-1+\phi/2}$, most spins will only be correlated at long distances through the weak perturbative effects mediated by already decimated spins. These involve many factors of the form $\tilde{h}/\tilde{J}$ (or $\tilde{J}/\tilde{h}$) at various energy scales down to $e^{-\Gamma}$. For spins with large separation, the smallest such factor will be of order $e^{-C\Gamma_L}\sim e^{-CL^{1/2}}$ yielding *typical critical correlations*

$$-\ln C_{ij}\sim|i-j|^{1/2}\,, \tag{4}$$

with a proportionality factor with a universal distribution with both width and mean of order unity. An equivalent result for transverse correlations was derived by Shankar and Murthy [2]. The *mean critical correlations*, on the other hand, behave very differently: The rare pairs of distant spins, i, j, which at some energy scale are both active in the same cluster, will have a correlation $C_{ij}\sim 1$ (reduced somewhat by the small length scale fluctuations) and dominate the mean. Thus $\overline{C}_{ij}\sim$prob(i and j active in the same cluster) yielding

$$\overline{C}_{ij}\sim|i-j|^{\phi-2}\,. \tag{5}$$

We now consider moving slightly away from the critical point, introducing $\delta\equiv\Delta-\Delta_c$. Linearizing the RG equations for R and P around the fixed point, one finds a single relevant eigenvalue $\lambda=1$, with $(R-R^*)=-(P$

$-P^*$), that corresponds to moving off the critical manifold. The RG flows will be roughly the same as at the critical point until $|\delta(\Gamma)| \sim 1$ which will happen at a logarithmic energy scale $\Gamma_\delta \sim 1/|\delta|$. Because the length scales at the critical point grow as $L_\Gamma \sim \Gamma^{-2}$, we thus find a *true correlation length* $\xi \sim |\delta|^{-\nu}$ with $\nu = 2$ which is therefore much larger than the typical correlation length $\tilde{\xi}$.

In the *disordered phase*, at a distance $\delta > 0$ from the critical point, the correlations C_{ij} between two typical spins separated by of order ξ will be $-\ln C_{ij} \sim \xi^{1/2}$ from the typical behavior at the critical point. Analysis of the RG flows in the disordered phase for $\Gamma \gg \Gamma_\delta$ shows that a reduction factor of order $e^{-\sqrt{\xi}}$ will occur in the long distance correlations for each distance of order ξ. Thus for $|i-j| \gg \xi$, we have *typical correlations* $-\ln C_{ij} \sim [|i-j|/\xi]\xi^{1/2} \sim |i-j|/\tilde{\xi}$ with $\tilde{\xi} \sim \xi^{1/2}$—a similar form to the transverse correlations [2].

The mean correlations again behave quite differently. To analyze these, one must find the distribution of *lengths* l of spin clusters at scale Γ. At a scale of order Γ_δ, this distribution is found to have an exponential tail of the form $\exp(-l/\Gamma_\delta^2) \sim \exp(-l/\xi)$. This tail dominates the probability that spins with separation $L \gg \xi$ will appear in the same cluster and thus be strongly correlated. As at the critical point, these strongly coupled pairs dominate the *mean correlations* yielding $\bar{C}_{ij} \sim \exp -|i-j|/\xi$ in the disordered phase.

Note that the typical correlation length exponent $\tilde{\nu} = 2$ violates the bound [6] for probabalistically defined finite-size scaling correlation lengths $\nu_{FS} \geq 2/d = 2$, while the exponent $\nu = 2$ for the mean correlations saturates the bound: Simple finite-size scaling lengths can be shown to be essentially equivalent to ξ [6]. Physically the behavior of mean correlations can be rationalized by considering the probability that at a distance δ above the critical point, a connected region containing both the spins i and j is, by a random fluctuation, effectively *below* its local approximate critical point. Since the variance of the $\Sigma_{ij} \equiv \sum_{k=i}^{j}(\ln J_k - \ln h_k)$ is of order $|i-j|^{1/2}$ and $\bar{\Sigma}_{ij} \sim -\delta L$ the probability that $\Sigma_{ij} > +\delta L$—which at least naively suggests strong correlations between S_i^z and S_j^z—is of order $\exp(-|i-j|^2\delta^2/|i-j|) \sim \exp(-|i-j|/\xi)$.

We now turn to the behavior in the *ordered phase*. At a length scale of order $\xi \sim |\delta|^{-2}$ at which $\delta(\Gamma) \approx -1$, the density of remaining active spins is of order $\Gamma_\delta^{\phi-2} \sim |\delta|^{\phi-2}$. At longer scales, the $\tilde{J}$'s will typically be larger than the $\tilde{h}$'s and so the spin clusters will be combined to form larger—and eventually infinite—clusters with only a few extra spins being decimated. Thus the *spontaneous magnetization* is determined at the scale Γ_δ yielding $m \sim |\delta|^\beta$ with $\beta = 2 - \phi = \frac{1}{2}(3-\sqrt{5}) \approx 0.38$, a new exact thermodynamic prediction. At length scales L larger than ξ, the effective J's continue to decrease (but only as $1/\ln L$); thus near the critical point there is no well-defined domain wall energy in an infinite chain. (Far enough into the ordered phase, in particular if *all* the J_j's are larger than *all* the h_j's, this is no longer true and the domain wall energy becomes nonzero [2].)

Our results can be generalized to include the effects of a uniform ordering field $H(\equiv H_z)$. For small H, we renormalize as in zero field until the magnetic energy of a spin cluster, $\tilde{g}H$, is of order the cutoff scale Ω. Near the critical point, this occurs when $\Gamma_H \sim \ln 1/|H|$. At lower energies and longer length scales, the uniform magnetic energy dominates, and all the remaining active spins get expectations of order unity. Thus we expect a scaling form for the magnetization near to the critical point,

$$m(H,\Delta) \approx |\delta|^\beta \operatorname{sgn}(H) \mathcal{M}\left(\delta \ln \frac{1}{|H|}\right), \tag{6}$$

with

$$m(H,\Delta_c) \sim \left(\ln \frac{1}{|H|}\right)^{-\beta} \tag{7}$$

at criticality and in the ordered phase $\mathcal{M}(u \to -\infty) \to \text{const}$. In the disordered phase, the moments of the spin clusters for $\Gamma_H \gg \Gamma \gg \Gamma_\delta$ remain of order $\tilde{g} \sim \Gamma_\delta^\phi$ while the lengths of the connecting bonds grow rapidly as $l_\Gamma \sim \xi \exp(k\delta\Gamma)$ with k a nonuniversal constant. Thus the magnetization in a small field is of order $\delta^{-\phi}/l_{\Gamma_H}$ yielding

$$m(H,\Delta) \sim |H|^{k\delta}, \tag{8}$$

i.e., a continuously variable power law. This arises, however, not from a fixed line, but from the form of the scaling function Eq. (6). [It is useful to note that a scaling function of a form similar to Eq. (6) is implied by McCoy's [2] calculations of the *surface magnetization* m_s as a function of a surface field H_s, differing from Eq. (6) primarily by the replacement of β by $\beta_s = 1$. This provides important support for the inexact derivation outlined here.]

We can now understand what happens to $m(H)$ as Δ increases from Δ_c: At some point, $\Delta = \Delta_\chi$, the exponent $\tilde{\delta} \approx k\delta + \cdots$ in Eq. (8) becomes equal to 1 and the linear susceptibility becomes finite, being no longer dominated by singular rare events. Above some higher value of Δ, the nonlinear susceptibility $\partial^3 m/\partial H^3$ also becomes finite, and finally, if the distribution of J's has an upper bound [and $\rho_0(h)$ a lower bound] $\tilde{\delta} \to \infty$ and the free energy becomes analytic for small h: This occurs at the upper boundary of the "Griffiths phase" [1,2], when *all* the J_j's are smaller than *all* the h_j's. Thus we see that the point Δ_χ is not very special; it is merely when the singular part of $m(H)$ dominates the linear part.

We finally turn to the behavior of the random transverse field chain at small positive temperatures T. An RG procedure similar to that outlined above for $H \neq 0$ can be used: Stop renormalizing at an energy scale $\Omega_0 e^{-\Gamma} \sim T$ [4]. At this point, the interactions between spin clusters will no longer force them into ground states, and we expect each cluster of scale Γ_T to act approximately like a free spin with magnetic moment $\tilde{g}$ ($\sim \Gamma_{\!\!\varphi}^{\phi}$ near Δ_c), Curie susceptibility, and entropy of order ln2. The linear susceptibility near Δ_c can then be shown [3] to have the form

$$\chi(T) \sim T^{-1+k\delta}, \tag{9}$$

as $T \to 0$ with the *same* coefficient k as in Eq. (8) and a $|\ln T|^{2\beta-2}$ factor appearing at Δ_c. The specific heat is similarly $C(T) \sim T^{k|\delta|}$ with symmetry about Δ_c implied by duality. At Δ_c, $C(T) \sim |\ln T|^{-3}$. Scaling functions with the scaling variable $\delta \ln T^{-1}$ exist both for thermodynamic quantities and mean spatial correlations.

We have seen that the simplest nontrivial random quantum system exhibits very peculiar behavior near to its zero-temperature quantum order-disorder transition. The logarithmic forms of energy-vs-length and energy-vs-$(\Delta - \Delta_c)$ scaling, as well as the extreme domination of mean correlations and thermodynamic quantities by rare events, are caused by what is effectively a fluctuationless, $\hbar = 0$, critical point. At small energy scales, each effective degree of freedom is either almost completely locally ordered (i.e., the spin clusters) or almost completely disordered (i.e., the decimated spins). The extreme dangerous irrelevancy of $\hbar$, which controls the quantum fluctuations, gives rise to the logarithmic scaling forms and other physics in an analogous way to activated dynamic scaling near the phase transition in classical random field magnets [5], with quantum tunneling here playing an analogous role to thermal activation in those systems.

The main question which one would like to understand is whether similar phenomena occur near other quantum phase transitions in random systems. In other 1D systems in which the randomness is important at long scales, similar physics will certainly occur both at and far from phase transitions [7]. In higher dimensions, the situation is much less clear. There are several natural possibilities: conventional scaling (as is believed to occur at metal-insulator [8] and superfluid-insulator [9] quantum transitions) with either the order parameter susceptibility diverging (a) at the transition or (b) before it [9], or (c) logarithmic scaling as in 1D with highly singular behavior at low temperatures near to the transitions: Although (b) can certainly occur, whether or not the most interesting possibility (c) exists in higher dimensions we leave as an intriguing open question.

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[1] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968); **188**, 982 (1969); *Theory of Two-Dimensional Ising Model* (Harvard Univ. Press, Cambridge, 1971), Chaps. 14 and 15; B. M. McCoy, Phys. Rev. **188**, 1014 (1969).
[2] R. Shankar and G. Murthy, Phys. Rev. B **36**, 536 (1987).
[3] D. S. Fisher (to be published).
[4] C. Dasgupta and S. K. Ma, Phys. Rev. B **22**, 1305 (1980).
[5] D. S. Fisher, J. Appl. Phys. **61**, 3672 (1987).
[6] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 299 (1986); Commun. Math. Phys. **120**, 501 (1989).
[7] For example, random *XXZ* spin chains, see Ref. [3].
[8] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[9] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).