Exactly Solvable Potentials and Quantum Algebras

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A set of exactly solvable one-dimensional quantum-mechanical potentials is described. It is defined by a finite-difference-differential equation generating in the limiting cases the Rosen-Morse, harmonic, and Pöschl-Teller potentials. A general solution includes Shabat's infinite number soliton system and leads to raising and lowering operators satisfying a q-deformed harmonic-oscillator algebra. In the latter case the energy spectrum is purely exponential and physical states form a reducible representation of the quantum conformal algebra $su_q(1,1)$.

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Lie algebras are among the cornerstones of modern physics. They have an enormous number of applications in quantum mechanics and, in particular, put an order in the classification of exactly solvable potentials. "Quantized," or q-deformed, Lie algebras (also loosely called quantum groups) are now well-established objects in mathematics [1]. Their applications were found in two-dimensional integrable models and systems on lattices. However, despite much effort quantum algebras do not yet penetrate into physics on a large scale. In this paper we add to this field and show that a q-deformed harmonic-oscillator algebra [2] may have straightforward meaning as the spectrum-generating algebra of the specific one-dimensional potential with exponential spectrum. This result shows that group-theoretical content of exactly solvable models is not bounded by the standard Lie theory.

Recently Shabat analyzed an infinite chain of reflectionless potentials and constructed an infinite number soliton system [3]. The limiting potential decreased slowly at space infinities and obeyed peculiar self-similar behavior. We will present corresponding results in slightly different notations. We denote the space variable by x and introduce N superpotentials $W_n(x)$ satisfying the following set of second-order differential equations:

$$(W_n'+W_{n+1}'+W_n^2-W_{n+1}^2)'=0,\quad n=0,\dots,N-1\,, \tag{1}$$

where primes denote derivative with respect to x. Taking first integrals

$$W_n'+W_{n+1}'+W_n^2-W_{n+1}^2=k_{n+1}\,, \tag{2}$$

where k_n are some constants, we define $N+1$ Hamiltonians

$$2H_n=p^2+U_n(x),\quad p\equiv -i\,d/dx\,,$$
$$U_0(x)=W_0^2-W_0'+k_0,\quad U_{n+1}(x)=U_n(x)+2W_n'(x)\,. \tag{3}$$

An arbitrary energy shift parameter k_0 enters all potentials $U_n(x)$.

Notorious supersymmetric Hamiltonians are obtained by unification of any two successive pairs H_n, H_{n+1} in a diagonal 2×2 matrix [4]. Analogous construction for the whole chain (3) was called an order N parasupersymmetric quantum mechanics [5,6]. In the latter case, relations (1) naturally arise as the diagonality conditions of the general $(N+1)\times(N+1)$-dimensional parasupersymmetric Hamiltonian. We do not use these algebraic constructions here and consider operators H_n on their own ground.

If $W_n(x)$'s do not have severe singularities then the spectra of operators (3) may differ only by a finite number of lowest levels. Under the additional condition that the functions

$$\psi_0^{(n)}(x)=\exp\left[-\int^x W_n(y)dy\right] \tag{4}$$

belong to the Hilbert space $\mathcal{L}_2$ one obtains the first N eigenvalues of the Hamiltonian H_0

$$H_0\psi_n^{(0)}(x)=E_n\psi_n^{(0)}(x),\quad E_n=\tfrac{1}{2}\sum_{i=0}^{n}k_i\,,\quad n=0,1,\dots,N-1\,, \tag{5}$$

where the subscript n enumerates levels from below. In this case (4) represents the ground-state wave function of H_n from which one can determine the lowest excited states of H_j, $j<n$, e.g., the eigenfunctions of H_0 are given by

$$\psi_n^{(0)}(x)\propto(p+iW_0)(p+iW_1)\cdots(p+iW_{n-1})\psi_0^{(n)}\,. \tag{6}$$

Any exactly solvable discrete spectrum problem can be represented in the forms (2)–(6). Sometimes it is easier to solve the Schrödinger equation by direct construction of the chain of associated Hamiltonians (3)—this is the essence of the so-called factorization method [7–9]. For the problems with only N bound states there does not exist $W_N(x)$ making $\psi_0^{(N)}$ normalizable. For example, if $W_N(x)=0$, then H_n has exactly $N-n$ levels, the potential $U_n(x)$ is reflectionless and corresponds to an $(N-n)$-soliton solution of the Korteweg–de Vries equation.

Let us consider potentials which support an infinite number of bound states, $N=\infty$. In this case one can derive from (2) the following differential equations involving only one derivative and a tail of W_n's:

$$W_i'(x)+W_i^2(x)+\sum_{j=1}^{\infty}(-1)^j[2W_{i+j}^2(x)+k_{i+j}]=0\,. \tag{7}$$

A question of convergence of the infinite sum is delicate and requires special consideration in each case. Evident condition $W_\infty(x)=W'_\infty(x)=0$, which is still related to the soliton dynamics, is necessary for rigorous justification of (7). Here we shall operate with a formal series and assume that the initial chain (2) may always be recovered by adding (7) for $i=n$ and $i=n+1$. In order to find an infinite number of superpotentials $\{W_i\}$ from (7) one has to relate them to one unknown function via some simple rule. Following Ref. [3] we take the ansatz

$$W_i(x)=q^iW(q^ix)\,, \tag{8}$$

which yields the equation

$$W'(x)+W^2(x)-\gamma^2+2\sum_{j=1}^{\infty}(-1)^jq^{2j}W^2(q^jx)=0\,, \tag{9}$$

where $\gamma^2=-\sum_{j=1}^{\infty}(-1)^jk_j$. Note that reality of superpotentials does not necessarily restrict the parameter q to be real—this will appear later. From (9) it is easy to derive Eqs. (7) and (2) with

$$k_{i+1}=\gamma^2(1+q^2)q^{2i},\quad i\geq 0\,. \tag{10}$$

The following computation

$$\gamma^2=-\sum_{j=1}^{\infty}(-1)^jk_j=\gamma^2(1+q^2)\sum_{j=1}^{\infty}(-1)^jq^{2j}\equiv\gamma^2 \tag{11}$$

shows that γ^2 is a completely arbitrary parameter (an energy scale) and (10) is self-consistent definition of the constants k_i. Derivation (11) is valid only at $|q|<1$, which was the restriction of Ref. [3], but if (8) and (10) are taken as the basic substitutes for (2) then by definition γ^2 is arbitrary and there are no essential restrictions on q up to now.

Equation (9) has a certain relation to quantum algebras [1] and corresponding q analysis [10]. In order to see this we first introduce a scaling operator T_q obeying the group law

$$T_qf(x)=f(qx),\quad T_qT_p=T_{qp}\,,$$
$$T_q^{-1}=T_{q^{-1}},\quad T_1=1\,. \tag{12}$$

Then (9) can be rewritten as

$$W'(x)-W^2(x)=\gamma^2-2\sum_{j=0}^{\infty}(-1)^j(q^2T_q)^jW^2(x)$$
$$=\gamma^2-2(1+q^2T_q)^{-1}W^2(x)\,. \tag{13}$$

Multiplying (13) from the left-hand side by $1+q^2T_q$ we obtain the finite-difference-differential equation defining $W(x)$,

$$W'(x)+W^2(x)+qW'(qx)-q^2W^2(qx)=\gamma^2(1+q^2)\,, \tag{14}$$

which is nothing else than the first iteration of superpotentials. The whole infinite chain (2) is thus generated by (14). This observation removes ambiguities arising in (9) due to the convergence problems.

Let us try to find the quantum-mechanical spectrum generated by the self-similar potential $U_0(x)$ associated with (14). Suppose that the eigenfunctions (4) are normalizable. Then the potential $U_{i+1}(x)$ contains one eigenvalue less than $U_i(x)$, i.e., there should be the following ordering of levels

$$E_0<E_1<\cdots<E_\infty\,,$$
$$E_n=\tfrac{1}{2}\sum_{i=0}^{n}k_i=-\tfrac{1}{2}\gamma^2\frac{1+q^2}{1-q^2}q^{2n}\,, \tag{15}$$

where we chose the undefined constant k_0 to be $k_0=-\gamma^2(1+q^2)/(1-q^2)$. At negative γ^2 it is not possible to fulfill the ordering and at positive γ^2 the parameter q should be real and lie in one of the regions $|q|<1$ or $|q|>1$. Taking the normalization $\gamma^2=\omega^2|1-q^2|/(1+q^2)$ and denoting $|q|=\exp(\pm\eta/2)$, $\eta>0$, we arrive at an exponentially small or large bound energy spectrum

$$E_n=\mp\tfrac{1}{2}\omega^2e^{\mp\eta n}\,. \tag{16}$$

What type of the potentials would these spectra correspond to? In order to know this one should solve Eq. (14). Then everything crucially depends on the normalizability of $\psi_0^{(0)}$ in (4) because all other wave functions $\psi_0^{(n)}$ are related to it by scaling. Normalizability is insured if $W(x)$ is a continuous function positive at $x\to+\infty$ and negative at $x\to-\infty$. Under such conditions $W(x)$ has at least one zero and we choose the corresponding point to be $x=0$, i.e., $W(0)=0$. Equation (14) now automatically leads to $W(-x)=-W(x)$ and below we restrict q to be semipositive. Let us find the solution of (14) in the Taylor series form near zero. Substituting an expansion $W(x)=\sum_{i=1}^{\infty}c_ix^{2i-1}$ into (9) we obtain the following recursion relation for the coefficients c_i:

$$c_i=\frac{q^{2i}-1}{q^{2i}+1}\,\frac{1}{2i-1}\sum_{m=1}^{i-1}c_{i-m}c_m,\quad i\geq 2,\quad c_1=\gamma^2\,, \tag{17}$$

which at $q=0$, $\gamma=1$ generates Bernoulli numbers B_{2i}, $c_i=2^{2i}(2^{2i}-1)B_{2i}/(2i)!$. One may say that (17) defines the q analogs of the Bernoulli numbers $[B_i]_q$. Equation (17) works well for all values of q. At $q<1$ it describes q deformation of the hyperbolic tangent, since at $q=0$ one has

$$W'+W^2=\gamma^2,\quad W(x)=\gamma\tanh\gamma x\,, \tag{18}$$

which is a one-level (soliton) superpotential associated to the Rosen-Morse problem. At $q>1$ one has q deformation of the trigonometric tangent which is recovered in the limit $q\to\infty$,

$$W'-W^2=\gamma^2,\quad W(x)=\gamma\tan\gamma x\,. \tag{19}$$

This superpotential creates an infinite-level Pöschl-Teller potential $U_1(x)$ with the restricted region of coordinate definition: $-\pi<2\gamma x<\pi$. On this finite cut $U_0(x)=0$ presents an infinitely deep potential well. If one sets $\gamma=0$

simultaneously with q or q^{-1} then conformal superpotentials, $W(x)=\pm 1/x$, are emerging. Finally, at $q=1$ one gets a standard harmonic-oscillator problem.

If $q\neq 0,1,\infty$, there is no analytical expression for $W(x)$ but some general properties of this function may be found along the analysis of Ref. [1], where it was proven that for $q<1$ superpotential is positive at $x=+\infty$. In this case the required normalizability condition is fulfilled and the relation (16) with upper signs really corresponds to physical spectrum.

At $q>1$ the radius of convergence of the series defining $W(x)$ is finite, $r_c<\infty$. From the inequalities

$$\frac{\gamma^2}{\omega^2}\equiv\frac{q^2-1}{q^2+1}<\frac{q^{2i}-1}{q^{2i}+1}<1,\quad i>1,$$

we have $0<c_i^{(1)}<c_i<c_i^{(2)}$, where $c_i^{(1,2)}$ are defined by rule (17) when the q factor on the right-hand side is replaced by γ^2/ω^2 and 1, respectively ($c_1^{(1,2)}=c_1$). As a result, $1<2\gamma r_c/\pi<\omega/\gamma$, which means that $W(x)$ is smooth only on some cut at the ends of which it has singularities. From the basic relation (14) it follows that there is an infinite number of simple "primary" and "secondary" poles. The former ones have residues equal to -1 and their location points x_m tend to $\pi(m+\frac{1}{2})/\gamma$, $m\in Z$, at $q\to\infty$. Secondary poles are situated at $x=q^n x_m$, $n\in Z^+$, and corresponding residues are defined by some algebraic equations. We are thus forced to consider Schrödinger operators (3) on a cut $[-x_1,x_1]$ and impose boundary conditions $\psi_n^{(i)}(\pm x_1)=0$ although the potential $U_0(x)$ is finite at $x=\pm x_1$. The structure of $W(x)$ leads to $\psi_0^{(0)}(\pm x_1)=0$; i.e., $\psi_0^{(0)}$ is the true ground state of H_0. Note, however, that the spectrum E_n for such type of problems cannot grow faster than n^2 at $n\to\infty$ in apparent contradiction with (16). This discrepancy is resolved by the observation that already $W_1(x)=qW(qx)$ has singularities in the interval $[-x_1,x_1]$ so that only H_0 and H_1 are isospectral in the chain (3). Hence, the positive sign case of (16) does not correspond to the real physical spectrum of the model.

The number of deformations of a given function is not limited. The crucial property preserved by the above presented q curling is the property of exact solvability of "undeformed" Rosen-Morse, harmonic-oscillator, and Pöschl-Teller potentials. It is well known that potentials at infinitely small and exact zero values of a parameter may obey completely different spectra. In our case, deformation with $q<1$ converts the one-level problem (18) with $E_0=-\gamma^2/2$ into an infinite-level one with exponentially small energy eigenvalues (16). Whether one gets an exactly solvable potential at $q>1$ is an open question but this is quite plausible because at $q=\infty$ a problem with the known spectrum $E_n=\gamma^2(n+1)^2/2$ arises.

In the standard dynamical symmetry approach the Hamiltonian of a system is supposed to be proportional either to the Casimir operator or to a polynomial combination of the generators of some Lie algebra [8,11]. As a result, the energy eigenvalues are determined by rational functions of quantum numbers. This means that one does not go out of the universal enveloping algebra. q deformation of the universal algebra works with functions (exponentials) of generators and, as was mentioned, accounts for the presented exponential spectra.

Indeed, substituting superpotentials (8) into relation (6) one finds the raising and lowering operators

$$\psi_{n\pm 1}^{(0)}\propto A^{\pm}\psi_n^{(0)},\quad H_0=\frac{1}{2}\left(A^+A^- - \frac{1+q^2}{1-q^2}\gamma^2\right),$$
$$A^+=q^{1/2}[p+iW(x)]T_q,\quad A^-=q^{-1/2}T_q^{-1}[p-iW(x)], \tag{20}$$

For real q and γ the operators $A^{\pm}$ are Hermitian conjugates of each other. Equation (14) insures the following q-commutation relations:

$$A^-A^+-q^2A^+A^-=\gamma^2(1+q^2),\quad H_0A^{\pm}=q^{\pm 2}A^{\pm}H_0. \tag{21}$$

Introduction of the formal number operator

$$N=\frac{\ln H_0/E_0}{\ln q^2},\quad N\psi_n^{(0)}=n\psi_n^{(0)},\quad [N,A^{\pm}]=\pm A^{\pm} \tag{22}$$

completes the definition of the q-deformed harmonic-oscillator algebra in the particular form [2]. The quantum conformal algebra $su_q(1,1)$ is realized as follows [12],

$$K^+=\left(\frac{q}{\gamma(1+q^2)}q^{-N/2}A^+\right)^2,$$
$$K^-=(K^+)^{\dagger},\quad K_0=\tfrac{1}{2}(N+\tfrac{1}{2}), \tag{23}$$
$$[K_0,K^{\pm}]=\pm K^{\pm},\quad [K^+,K^-]=-\frac{q^{4K_0}-q^{-4K_0}}{q^2-q^{-2}},$$

i.e., it is a dynamical symmetry algebra of the model. Generators $K^{\pm}$ are parity invariant and therefore even and odd wave functions belong to different irreducible representations of $su_q(1,1)$.

In order to generalize the basic equation (14) we introduce an additional parameter s into the superpotential, $W=W(x,s)$, and assume that T_q in (20) is a generalized shift operator,

$$T_qW(x,s)=W(qx+a,s+1), \tag{24}$$

where q and a are parameters of affine transformation. Although A^+ is not Hermitian conjugate of A^- anymore we force them to obey q-oscillator-type algebra

$$A^-A^+-q^2A^+A^-=C(s),\quad A^{\pm}C(s)=C(s\pm 1)A^{\pm},$$

where C is some function of s. The resulting equation for the superpotential

$$W'(x,s-1)+qW'(qx+a,s)+W^2(x,s-1)-q^2W^2(qx+a,s)=C(s) \tag{25}$$

may be called the generalized shape-invariance condition (cf. [9]).

We define a Hamiltonian H as follows:

$$H=\frac{1}{2}A^+A^-+F(s),\quad q^2F(s)-F(s-1)=\frac{1}{2}C(s)\,, \tag{26}$$

where the finite-difference equation for the function $F(s)$ is found from the braiding relations $HA^{\pm}=q^{\pm 2}A^{\pm}H$. Now it is easy to generalize formula (15). Suppose that a wave function ψ_0, $A^-\psi_0=0$, is normalizable. Then a tower of higher states $\psi_n\propto(A^+)^n\psi_0$ gives the energy spectrum

$$E_n=F(s)+\frac{1}{2}\sum_{i=1}^{n}q^{2(i-1)}C(s+i)=q^{2n}F(s+n)\,, \tag{27}$$

which can be found by purely algebraic means. If for some $n=N$ normalizability of ψ_n is broken then H has only N discrete levels. In the above presentation we chose the simplest form of s-parameter transformation under the action of the T_q operator. One can easily generalize formula (27) for an arbitrary change of the variable s in (24), $s\to f(s)$.

To conclude, in this paper we have described an exactly solvable quantum-mechanical problem where the quantum algebra $su_q(1,1)$ acts on the discrete set of energy eigenstates scaling their eigenvalues by a constant factor. In the original version of differential geometric applications of quantum Lie algebras an underlying space was taken to be noncommutative ("quantum plane") and deformation parameter q was measuring deviations from normal analysis (see, e.g., Ref. [13]). Here we have commutative space and standard one-dimensional quantum mechanics but the potential is very peculiar. It represents q deformation of exactly solvable potentials so that the spectrum remains to be known but it acquires essentially functional character.

It is interesting to know the most general exactly solvable q-deformed potential. One of the approaches to this problem consists in repetition of the manipulation described in Ref. [14]. Namely, one can take as the particle's wave function a q-hypergeometric function multiplied by some weight factor. This would correspond to the transformation of q-hypergeometric equation to the form of the standard Schrödinger equation for some potential. Another path to q deformation of known models is given by Eq. (25) which may have solutions generalizing those found by the old factorization technique at $q=1$, $a=0$.

Two final remarks are in order. First, affine transformations appearing in (25) may be used for the definition of the q-deformed supersymmetric quantum mechanics [15]. Second, at complex values of q one has meaningful dynamical systems which are exactly solvable when q is a root of order unity [16].

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[1] V. G. Drinfeld, Sov. Math. Dokl. **32**, 254 (1985); M. Jimbo, Lett. Math. Phys. **10**, 63 (1985); **11**, 247 (1986); N. Yu. Reshetikhin, L. A. Takhtajan, and L. D. Faddeev, Algebra Analiz, **1**, 178 (1989).
[2] L. C. Biedenharn, J. Phys. A **22**, L873 (1989); A. J. Macfarlane, J. Phys. A **22**, 4581 (1989).
[3] A. Shabat, Inverse Prob. **8**, 303 (1992).
[4] E. Witten, Nucl. Phys. **B188**, 513 (1981).
[5] V. A. Rubakov and V. P. Spiridonov, Mod. Phys. Lett. A **3**, 1337 (1988); V. Spiridonov, in *Proceedings of the Twentieth DGM Conference, New York, 1991*, edited by S. Catto and A. Rocha (World Scientific, Singapore, 1992), p. 622.
[6] S. Durand, M. Mayrand, V. P. Spiridonov, and L. Vinet, Mod. Phys. Lett. A **6**, 3163 (1991).
[7] L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1951).
[8] W. Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968).
[9] L. E. Gendenstein, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 299 (1983) [JETP Lett. **38**, 356 (1983)].
[10] For a review and list of references, see R. Floreanini and L. Vinet, in Proceedings of the Second International Wigner Symposium, Goslar, Germany, 1991 (to be published).
[11] Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **148**, 346 (1983); M. Moshinsky, C. Quesne, and G. Loyola, Ann. Phys. (N.Y.) **198**, 103 (1990).
[12] P. P. Kulish and E. V. Damaskinsky, J. Phys. A **23**, L415 (1990).
[13] J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) **18**, 302 (1990); B. Zumino, Mod. Phys. Lett. A **6**, 1225 (1991).
[14] G. A. Natanzon, Vestn. Leningr. Univ. **10**, 22 (1971); O. V. Bychuk and V. P. Spiridonov, Mod. Phys. Lett. A **5**, 1007 (1990).
[15] V. Spiridonov, Mod. Phys. Lett. A **7**, 1241 (1992).
[16] A. Shabat and V. Spiridonov (unpublished).