Suppressing Chaos in Neural Networks by Noise

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We study discrete parallel dynamics of a fully connected network of nonlinear elements interacting via long-range random asymmetric couplings under the influence of external noise. Using dynamical mean-field equations, which become exact in the thermodynamical limit, we calculate the activity and the maximal Lyapunov exponent of the network in dependence of a nonlinearity (gain) parameter and the noise intensity.

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Recently there has been considerable interest in spatially extended dynamical systems such as neural networks [1–3], ecological and economical models [4], and populations of interacting oscillators [5]. Such systems may be described by a set of coupled differential equations or iterated maps.

In this Letter we show that studying these systems with parallel discrete dynamics has several advantages as compared to the continuous time treatment. For parallel discrete dynamics the mean-field equations (MFE) simplify such that explicit solutions can be obtained for special cases. One could then use perturbation theories to explore new properties of the system. In addition, our formalism is suited in a natural way for Monte Carlo simulations of the MFE [6].

As an example we study the transition to chaos in a random neural network with infinite-range interactions, which has been studied previously by Sompolinsky, Crisanti, and Sommers [2] in the continuous time limit. It will be shown that with our method the previous results can be reproduced in much simpler form and new results concerning the dependence of the largest Lyapunov exponent on noise can be obtained. We note that in general results obtained with discrete parallel dynamics need not coincide with those obtained for sequential updating or continuous behavior [3, 7].

Our model consists of N analog neurons $\{S_i(t)\}$, $i = 1,\ldots,N$, with $-1 \le S_i \le 1$, where every neuron S_i is connected to all other neurons S_j by couplings J_{ij}, which are uncorrelated Gaussian random variables with zero mean and variance $[J_{ij}^2]_J = 1/N$, where square brackets denote the "quenched" average over the couplings. We use parallel dynamics for updating of the neurons:

$$S_i(t+1) = \phi(h_i(t)), \quad i = 1,\ldots,N. \tag{1}$$

The sigmoidal transfer function $\phi(h)$ is assumed to be odd [i.e., $\phi(-h) = -\phi(h)$] and approaches ± 1 in the limit $h \to \pm\infty$. Furthermore, $\phi(h)$ should increase in the neighborhood of $h = 0$ as

$$\left.\frac{\partial}{\partial h}\phi(h)\right|_{h=0} = g, \tag{2}$$

where $g > 0$ is a gain parameter. The internal field of the neuron S_i is given by

$$h_i(t) = \sum_{j\neq i} J_{ij} S_j(t) + \xi_i(t), \tag{1$'$}$$

with ξ_i as external white noise of zero mean and variance

$$\langle \xi_i(t)\xi_j(\tau)\rangle = \sigma^2 \delta_{ij}\delta_{t\tau}, \tag{3}$$

where the angular brackets denote the "thermal" average over the noise with intensity σ^2.

To study the dynamical properties of the network we start with the dynamical functional approach using the fact that averaged dynamical quantities can be obtained by differentiation from the following generating function:

$$[Z(l)]_J = \left[\int_{-\infty}^{+\infty} \prod_{i,t} \frac{dh_i(t)\, d\hat{h}_i(t)}{2\pi} \exp\left\{-\frac{\sigma^2}{2}\sum_{i,t}\hat{h}_i^2(t) - \sum_{i,t} i\hat{h}_i(t)h_i(t) + \sum_{i\neq j} J_{ij}\sum_t i\hat{h}_i(t)S_j(t) + \sum_{i,t} l_i(t)S_i(t)\right\}\right]_J . \tag{4}$$

We remark that the normalization is $[Z(0)]_J \equiv 1$. After some calculation following [8] the dynamics of the whole system can be reduced to an equation for an effective single neuron in the thermodynamical limit:

$$S(t+1) \equiv S_{t+1} = \phi(h_t). \tag{5}$$

The internal field h_t is now a self-consistent time-dependent Gaussian field with zero mean obtained from the equations for the unique saddle point:

$$\begin{aligned}\langle h_t h_\tau\rangle &= \sigma^2\delta_{t\tau} + \langle S_t S_\tau\rangle \\ &= \sigma^2\delta_{t\tau} + \langle \phi(h_{t-1})\phi(h_{\tau-1})\rangle.\end{aligned} \tag{6}$$

Studying (5) and (6) leads now to the same averaged dynamical properties as (1). Thus we can define the activity of the network,

$$K_t = \langle h_t^2 \rangle = \sigma^2 + \langle \phi^2(h_{t-1}) \rangle = \sigma^2 + \int_{-\infty}^{+\infty} Dx\ \phi^2(\sqrt{K_{t-1}}x), \tag{7}$$

with the abbreviation $Dx \equiv dx(2\pi)^{-1/2}\exp\{-x^2/2\}$. Evaluating (7) leads to fixed points K^* as illustrated in Fig. 1. For small K_{t-1} one gets

$$K_t = \sigma^2 + g^2 K_{t-1} + O(K_{t-1}^2). \tag{8}$$

The trivial fixed point $K^* = 0$ exists only in the noiseless case and is stable for $g < 1$. For $\sigma^2 > 0$ or $g > 1$ we find only one stable fixed point with nonzero activity $0 < K^* \leq 1 + \sigma^2$.

To find the chaotic region we build a replica of our system with infinitesimal different initial conditions and the same noise $\xi_i^1(t) = \xi_i^2(t)$. Now we study the correlation between the two replicas using the method of generating function. The equations for the replicated effective neurons are

$$\begin{aligned} S_{t+1}^\alpha &= \phi(h_t^\alpha), \\ \langle h_t^\alpha h_\tau^\beta \rangle &= \sigma^2 \delta_{t\tau} + \langle S_t^\alpha S_\tau^\beta \rangle \\ &= \sigma^2 \delta_{t\tau} + \langle \phi(h_{t-1}^\alpha)\phi(h_{\tau-1}^\beta) \rangle, \end{aligned} \tag{9}$$

with $\alpha, \beta = 1, 2$. The maximal Lyapunov exponent is now derived from

$$\lambda = \lim_{\tau\to\infty} \lim_{S_t^1 \to S_t^2} \frac{1}{2\tau} \log_2 \frac{\langle (S_{t+\tau}^1 - S_{t+\tau}^2)^2 \rangle}{\langle (S_t^1 - S_t^2)^2 \rangle}. \tag{10}$$

Indeed the infinitesimal formulation is equivalent to the definition by a perturbation flow in the original N-particle picture [9]. Assuming equilibrium

$$K = K^* = \langle (h_t^1)^2 \rangle = \langle (h_t^2)^2 \rangle \tag{11}$$

and defining the correlation between the replicated neurons

$$C_t = \langle h_t^1 h_t^2 \rangle = \sigma^2 + \langle S_t^1 S_t^2 \rangle = \sigma^2 + \int_{-\infty}^{+\infty} Dx\ Dy\ \phi(\sqrt{K}x)\phi\left(\frac{C_{t-1}}{\sqrt{K}} x + \sqrt{\frac{K^2 - C_{t-1}^2}{K}} y \right), \tag{12}$$

Eq. (10) can be reduced to

$$\lambda = \frac{1}{2} \log_2 \left. \frac{\partial C_t}{\partial C_{t-1}} \right|_{C_{t-1}=K} = \frac{1}{2} \log_2 \int_{-\infty}^{+\infty} Dx\ [\phi'(\sqrt{K}x)]^2. \tag{13}$$

For the noiseless case and $g \leq 1$, where the system is in the trivial fixed point $K^* = 0$, we thus find

$$\lambda = \log_2 g. \tag{14}$$

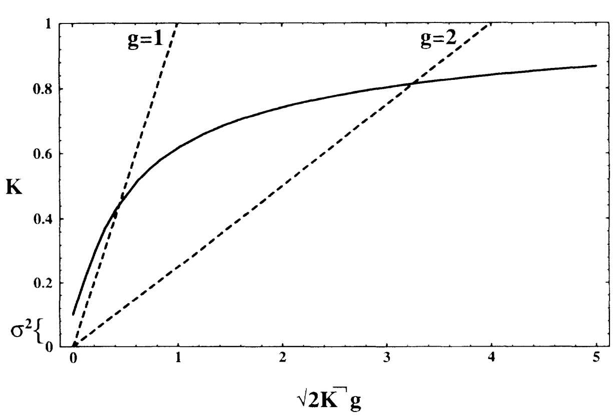

FIG. 1. Graphical solution for the equilibrium activity determined by the fixed point equation (7). The convex curve represents the right-hand side of the equation and the dashed lines the left-hand side for $g = 1, 2$. Noise shifts the curve to higher values as indicated by the finite value at $K = 0$.

We should note here that this result for the stability of the trivial fixed point can be obtained easily by using the elliptic law for the spectrum of eigenvalues of a random matrix [10] without using the method of generating function.

This system shows chaos for $g > 1$ in the noiseless case with the asymptotic behavior $\lambda = \frac{1}{2}\log_2$ in the $g \to \infty$ limit for every finite noise and every transfer function with the above described characteristic.

For further discussion we choose for concreteness

$$\phi(h) = \begin{cases} -1 & \text{for } h < -1/g, \\ gh & \text{for } -1/g \leq h \leq +1/g, \\ +1 & \text{for } +1/g < h\,. \end{cases} \tag{15}$$

To calculate the Lyapunov exponent for a certain noise and gain parameter we have to find the fixed point K^* by evaluating (7) and inserting in (13). In general this has to be done numerically as shown in Fig. 2. In the limit $g \to \infty$ we get

$$\begin{aligned} K^* &= 1 + \sigma^2, \\ \lambda &= \frac{1}{2} \log_2 \sqrt{\frac{2}{\pi K^*}}\, g. \end{aligned} \tag{16}$$

Hence the asymptotic behavior is $\lambda = \frac{1}{2}\log_2 g$, as expected.

The behavior of the system under the influence of noise is changed drastically. Generally the activity K^* of the network is increased. For $g \lesssim g_c$, the activity rises strongly in a nonlinear fashion with the onset of noise. This may be seen in Fig. 1. With increasing noise the

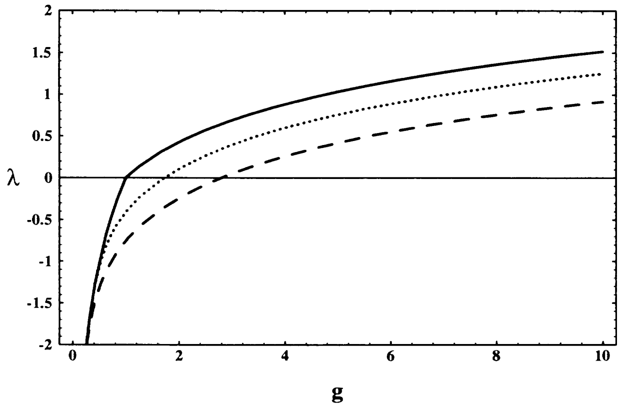

FIG. 2. The largest Lyapunov exponent as a function of the gain parameter for $\sigma = 0$ (solid line), $\sigma = 1$ (dotted line), and $\sigma = 2$ (dashed line). The sharp bend at $g = 1$ in the noiseless case indicates the phase transition to nonzero activity.

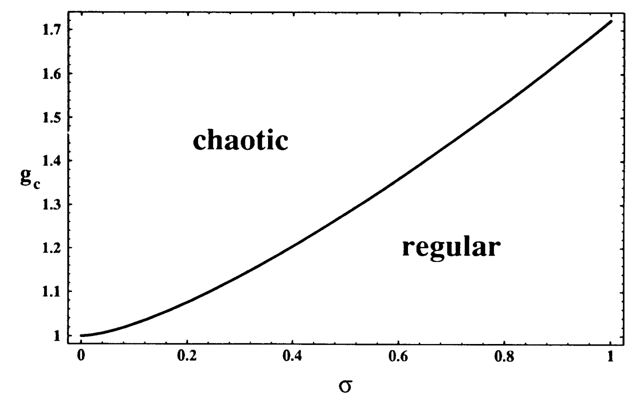

FIG. 3. Phase diagram for an infinite-range neural network with asymmetric couplings and external white noise. For a given gain parameter g chaos is suppressed by sufficient large amplitude of the noise σ.

critical gain parameter g_c, where chaos sets in, is shifted to larger values as shown in Fig. 3. For small noise we find the asymptotic behavior $g_c = 1 - \sigma^2 \ln \sigma^2$ and for large noise $g_c = \sqrt{\pi/2}\,\sigma$.

This depression of chaos is different from what is observed in low dimensional systems where chaos is favored by external noise [11, 12]. In high dimensional systems where chaos is due to interactions between nonchaotic elements, noise impairs the information flow between these elements and therefore tends to suppress chaos; i.e., destruction of chaos by external noise seems to be a fairly general property of such systems.

We have solved the dynamical MFE for a fully connected network of McCulloch-Pitts neurons with random interactions and computed its phase diagram in the presence of external noise (see Figs. 2 and 3). Our results show that the MFE simplify considerably for discrete parallel dynamics as compared to the continuous time model.

Therefore we expect that our method could also be applied to other spatially extended nonlinear systems. Some examples would be neural networks or oscillator systems with asymmetric couplings or time-delayed interactions. In a forthcoming paper we will investigate the dependence of the largest Lyapunov exponent of a neural network on the symmetry and time delay of the interactions using Monte Carlo simulations for the MFE following [6]. In addition, perturbation in the symmetry parameter similar to [13] should be possible.

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