The Shape Eigenstate: A New Kind of Resonance

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A new type of resonance appears to have been discovered in the $^{12}C(^{12}C,^{12}C(0_2^+))^{12}C(0_2^+)$ reaction at $E_{c.m.}=33$ MeV. We suggest that it is formed coherently from nearly degenerate resonances with different l values. Taking the individual resonances to be members of a cluster rotational band with $2n+1=$const, the coherence can be shown to follow from Levinson's theorem. The new kind of resonance does not have a definite angular momentum but corresponds to an approximate shape eigenstate.

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In some of the first studies of the collisions of two complex nuclei one of the unexpected features was the existence of narrow resonance in the yield of γ rays and light ions [1]. These experiments studied the collision of two ^{12}C nuclei but in later experiments resonances were found to occur in many heavy-ion collisions, and similar results have been found in the scattering of identical nuclei as heavy as ^{28}Si. Resonance phenomena have also been found in the collisions of nonidentical nuclei [2]. One of the best studied systems is $^{12}C+^{12}C$, and resonances have been discovered at center of mass energies up to and in excess of 30 MeV [3].

Most resonances studied [2] appear to be eigenstates characterized by a single angular momentum value, and this implies that the compound system cannot have a definite direction in space. Although two such resonances may overlap, this is quite a rare situation which has been successfully understood in the past in terms of band crossings [4] or the double resonance mechanism [5]. However, in heavy-ion scattering it might be possible to observe the effects of another quite novel kind of resonance in which many resonances with different angular momenta are excited simultaneously.

Data have recently been published for the $^{12}C(^{12}C,^{12}C(0_2^+))^{12}C(0_2^+)$ reaction [6] which strongly suggest the excitation of a 6α-particle-chain state in ^{24}Mg as predicted by α-cluster-model calculations [7]. This identification is made by supposing that the observation of two $^{12}C(0_2^+)$ 3α-chain states [8,9] in the exit channel arose from the fission of a ^{24}Mg 6α-chain state. One of the most striking features of these data is the behavior of the cross section around 90° in the center-of-mass frame (see Fig. 1). The cross section on resonance is very strongly enhanced near 90°. In this Letter we present an explanation of this unexpected behavior, in terms of overlapping resonances.

Physically, if we think of a deformed compound nucleus being created in such circumstances, the existence of overlapping resonances could produce a density distribution for the compound system with a definite orientation in the laboratory. For definiteness we will assume that this recently discovered resonance does indeed correspond to the 6α configuration. However, our arguments do not depend on the detailed structure of the compound system, only that there are many overlapping resonances. We concentrate here on the angular dependence of the state in space and the angular distribution of the decay fragments. We also show that some information can be derived on the spatial extension. The simple wave function that we arrive at can be regarded as a partial realization of an eigenstate of the angular position operator, $\cos\theta$.

The 6α-chain state resonances with different angular momenta have been predicted to be very closely spaced [7]. When the widths are large compared to the spacing the resonances can be considered degenerate. Such a set of resonances can be described by a cluster model in which the Wildermuth condition $2n+l=G$ (=const) is satisfied [10,11] and we predict that the angular distribution for this reaction should show a strong enhancement

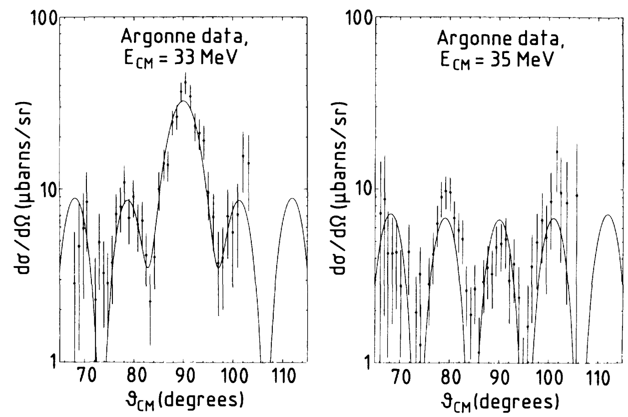

FIG. 1. Measured angular distributions for the $^{12}C(^{12}C,^{12}C(0_2^+))^{12}C(0_2^+)$ reaction [1] at 33 MeV (on resonance) and 35 MeV (off resonance). The 33-MeV data are fitted using Eq. (9) with $l_{max}=18$ and $A^2=0.15905$ μb/sr (and $B^2=0.18644$ μb/sr and $L=16$ for the incoherent background term). The 35-MeV data are presented with a squared Legendre polynomial of order 16 to guide the eye. The contrast between the two data sets emphasizes the unusual nature of the 90° peak on resonance.

at 90°, on resonance, in good agreement with the data. The result follows from consideration of the phase differences between resonances in a cluster rotational band and is a consequence of Levinson's theorem [12]. Indeed the phases are such that the data can be interpreted in terms of alignment of the 6α-chain state perpendicular to the beam axis. Hence we might consider these data as evidence of a new type of resonance—a resonance which is not an eigenstate of angular momentum, but rather an approximate "shape" eigenstate, built from a superposition of many partial waves.

It is no surprise that the angular distribution for the scattering of two spin-0 bosons should peak at 90°. However, it is rather more mysterious that the 90° maximum should be some 4–5 times greater than the neighboring local maxima. This can certainly not be achieved with the square of a single Legendre polynomial, which immediately suggests the necessity for interference of several Legendre polynomials. If we write (with even l only)

$$f(\theta)=\sum_{l=l_{\min}}^{l_{\max}}(2l+1)a_l P_l(\cos\theta) \tag{1}$$

and note that

$$P_l(\cos(\pi/2))=\begin{cases}(-1)^{l/2}\dfrac{l!}{2^l[(l/2)!]^2} & \text{for } l \text{ even}\\ 0 \text{ for } l \text{ odd}\end{cases} \tag{2}$$

we see that a_l must be proportional to $(-1)^{l/2}$ for constructive interference to be produced at 90°.

We now outline the steps necessary to determine the phase when the individual resonances excited in the exit channel all belong to a cluster-model rotational band characterized by a constant value of $2n+l$ (recall that n is the number of internal nodes in the radial wave function). We aim to highlight the essential features rather than perform a fully rigorous calculation for comparison with the experimental data. To do this we make several assumptions: First, we assume that there are no resonances in the entrance channel, so that the entrance channel phase shift is not varying rapidly; second, we assume that the coupling between the entrance and exit channels is weak, so that the distorted wave Born analysis (DWBA) approximation is valid (in particular, we require that the resonances in the exit channel do not significantly influence the entrance channel). For the moment, we ignore all Coulomb and absorption effects in the entrance and exit channels. Each partial DWBA amplitude for a monopole transition (with all spins zero) can be written (apart from numerical constants) as

$$A_l=\int \psi_f^l(r)V(r)\psi_i^l(r)r^2dr\,, \tag{3}$$

where $\psi_i^l(r)$ and $\psi_f^l(r)$ are elastically scattered waves in the incident and final channels, having angular momentum l, and $V(r)$ is the nuclear transition potential, assumed real. We shall show that neglecting absorption effects, the above matrix element takes the approximate form $A_l\sim\exp(i\delta_f^l)\exp(i\delta_i^l)|A_l|$, where δ_i^l and δ_f^l are the elastic scattering phase shifts in the entrance and exit channels. In this form the DWBA matrix element looks very similar to that used in the eikonal approximation [13] where $|A_l|$ would be related to a plane wave matrix element. The same form is obtained in semiclassical approximations when the S matrix for a transition from channel i to channel f is taken to be proportional to the geometric mean of the S matrices for elastic scattering in those two channels [14].

The solutions of the radial wave equation for the scattering in the exit channel must be regular at the origin where we define them to be real with the form $\lim_{r\to 0}\phi_l(r)\sim +r^{l+1}$. This follows if the scattering potential is taken to be real. However, this takes no account of the necessity to satisfy the asymptotic boundary conditions. Following the sequence of steps in Ref. [15] the physical wave function $\psi_l(r)$ which satisfies the asymptotic boundary conditions can be shown to be related to ϕ_l by

$$\psi_l(k,r)=\frac{k^{(l+1)}\phi_l(k,r)}{|F_+^l(k)|(2l+1)!!}\exp(i\delta_l)\,, \tag{4}$$

where the dependence of ψ_l and ϕ_l on k is indicated explicitly, and the Jost function has been introduced in modulus/argument form as $F_+^l(k)=+|F_+^l(k)| \times\exp[-i\delta_l(k)]$. The phase of $\psi_i^lV(r)\psi_f^l$ then follows from Eq. (4),

$$\psi_i^l(k,r)V(r)\psi_f^l(k,r)=\left[\frac{k^{2(l+1)}\phi_i^l(k,r)V(r)\phi_f^l(k,r)}{[(2l+1)!!]^2|F_{i+}^l(k)||F_{f+}^l(k)|}\right] \times\exp(i\delta_i^l+i\delta_f^l)\,. \tag{5}$$

Here the term in square brackets is real, and on substitution in Eq. (3) we obtain the desired result. We now proceed to use this result together with our first basic assumption that δ_i^l is slowly varying with l so that the l dependence in the phase of the matrix elements is dominated by $\exp(i\delta_f^l)$. Although the entrance channel phase should certainly vary with l, we assume this to be negligible compared with the contribution from the exit channel.

Next we invoke Levinson's theorem [12] which states that the phase shift is continuous and uniquely defined for each l value and that

$$\delta_l(E=0)-\delta_l(E\to\infty) =\begin{cases}\pi(N_l+\frac{1}{2}) \text{ if } l=0 \text{ and } F_+^l(0)=0,\\ \pi N_l \text{ otherwise}\,,\end{cases} \tag{6}$$

where N_l is the number of bound states with angular momentum l. The special case of $l=0$ and $F_+^l(0)=0$ refers to s-wave resonances at zero energy and is physically very rare. The final step in obtaining a definite phase relation between different partial waves is to combine these very general considerations with a cluster-model interpretation of the excited states in ^{24}Mg. We have previously argued [16] that a ^{24}Mg 6α-chain state, considered as two 3α-chain states end to end, should have $2n+l=G=36$ (where n is the number of nodes, inside the potential, of the radial wave function describing the relative motion, and l is the angular momentum). The precise value of $2n+l$ is not important so long as it remains constant within the band and is large enough to allow breakup into two ^{12}C(0_2^+) states with $l \gtrsim 18$. The association of the 0_2^+ state of ^{12}C with a 3α-chain state is a long-standing identification [8,9].

Consider now nonzero values of l. We denote by $N_l^b - 1$ the number of internal nodes in the wave function of the N_l^bth and last bound state, of angular momentum l. As we increase the energy above threshold we introduce an extra phase shift of π and one extra node inside the potential as we cross each successive l resonance. The first resonance occurs at a phase shift $N_l^b\pi+\pi/2$. Therefore we can generally say that the phase shift for partial wave l at the center of a resonance of energy E is $(N_l^b+N_l^r+\frac{1}{2})\pi$, where N_l^b is the number of bound states and N_l^r the number of resonant states of angular momentum l lying below E_l. Now for a cluster band characterized by a fixed value of $2n+l=G$, we can relate the number of nodes to the angular momentum (remember that $N_l^b+N_l^r=n-1$) and write (for nonzero l) $\delta_l(E)=\frac{1}{2}(G-l-1)\pi$. This immediately yields the required result, $\delta_l(E)=\text{const}-l\pi/2$.

We now employ this crucial phase relation to fit the measured angular distributions. Here we seek to reproduce the 33-MeV data of Wuosmaa *et al.* [6] on resonance with the simple form

$$\frac{d\sigma}{d\Omega}=\left|\sum_{l=l_{\min}}^{l_{\max}} e^{-i\pi l/2}(2l+1)a_l P_l(\cos\theta)\right|^2 + B^2|(2L+1)P_L(\cos\theta)|^2, \tag{7}$$

where we have added a background term to account for any nonresonant contributions. This background term is assumed to be 90° out of phase on resonance so that its contribution can in effect be added incoherently to that of the resonant term. The amplitudes a_l are now real and positive, and $e^{-i\pi l/2}$ includes the complex phase to within a constant (including the entrance channel phase). We present details of the energy variation of the cross section on passing through resonance elsewhere [17]. In Ref. [17] we tacitly assume that all a_l's have an identical Breit-Wigner energy dependence. The best fit to the published data [6] is obtained with $E_0=32.8$ MeV and width $\Gamma=4.7$ MeV. The quality of the fits obtained in Ref. [17] is comparable to that presented here.

We have fitted the data with various l-dependent forms of the amplitudes a_l. We find that we obtain the best fits to the data (in a minimum χ^2 sense) if we take $l_{\min}\approx 2$ and assume that the amplitudes a_l vary slowly with l, but the low partial waves $l=0$–4 are not essential to get an adequate fit. In particular, if we choose

$$e^{-i\pi l/2}a_l=(-1)^{l/2}\frac{Al!}{2^l[(l/2)!]^2} \approx (-1)^{l/2}A\left(\frac{2}{\pi l}\right)^{1/2} \quad \text{for } l>0\,, \tag{8}$$

and remembering that only even-l terms occur in the sum, an excellent fit is obtained with $l_{\min}=0$. This choice for a_l enables us to use Eq. 8.9.1 of Ref. [18] to obtain a simple closed form expression for the differential cross section,

$$\frac{d\sigma}{d\Omega}=A^2\left[\frac{(l_{\max}+1)!}{2^{l_{\max}}[(l_{\max}/2)!]^2}\frac{P_{l_{\max}+1}(\cos\theta)}{\cos\theta}\right]^2 + B^2|(2L+1)P_L(\cos\theta)|^2\,, \tag{9}$$

and A^2 has a Breit-Wigner energy dependence. Figure 1 shows the 33-MeV data fitted with the above form taking $l_{\max}=18$ and $A^2=0.15905$ μb/sr for the resonant term and $L=16$ and $B^2=0.18644$ μb/sr for the background term. With these values we obtain a χ^2 of 0.99 per degree of freedom. The value of $l_{\max}$ required by the fit is close to the experimental grazing angular momentum and is basically determined by the width of the 90° peak. This indicates that the chain state rotational band has been excited up to approximately the grazing l and that most of the partial waves contribute coherently to the resonant cross section. The background term has $L=16$ which is the dominant grazing angular momentum [6], as would be expected for a nonresonant peripheral reaction mechanism.

A simple physical interpretation of the 90° peak can be given in terms of the quantal deflection function. Here we use $\Theta=\partial(\delta_l^i+\delta_l^f)/\partial l$, where δ_l^i and δ_l^f are the phase shifts in the entrance and exit channels, respectively. We assume that the phase shift in the entrance channel is independent of l, so that Θ builds up only in the exit channel. The value of Θ is found by differentiating the phase relation obtained from the Wildermuth condition with respect to l, so that $\partial\delta_l^f/\partial l=-\pi/2$, with our assumption about δ_l^i. This implies that particles come in along the beam direction and are predominantly scattered at 90°.

A more intuitive insight into the physical situation is provided by supposing that a very deformed axially symmetric $K=0$ nucleus, oriented in space at the polar angle θ_0 is formed in the reaction. This undergoes symmetric fission with the fragments emitted at the polar angle θ. There is no information on the azimuthal angles, so we know only that the symmetry axis of the nucleus is distributed on a cone of angle θ_0, and that the fission fragments are to be found on a similar cone of angle θ. Consider a limiting case in which any change of the polar angle of the deformed nucleus produces a state orthogonal to the initial state. Mathematically this states that $\langle\chi(\cos\theta)|\chi(\cos\theta_0)\rangle=(1/2\pi)\delta(\cos\theta-\cos\theta_0)$. The azimuthal angle is completely undetermined above. Let us now expand the δ function $\delta(\cos\theta-\cos\theta_0)$ in terms of the complete orthogonal set of Legendre polynomials,

$$\delta(\cos\theta-\cos\theta_0)=\sum_l a_l P_l(\cos\theta)\,. \qquad (10)$$

Using the orthogonality of the Legendre functions we may determine the expansion coefficients as $a_l = \frac{1}{2}(2l+1)P_l(\cos\theta_0)$. We thus obtain

$$\delta(\cos\theta-\cos\theta_0)=\frac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta_0)P_l(\cos\theta)\,. \qquad (11)$$

If we now choose $\theta_0=90°$, we obtain the result that the odd partial waves vanish, since $P_{l\,\mathrm{odd}}(90°)=0$, and when l is even we get a phase $e^{i\pi l/2}=(-1)^{l/2}$ multiplying $P_l(\cos\theta)$ [see Eq. (2)]. This expansion of the δ function is identical in form to Eqs. (7) and (8) except that the sum runs over even values from $l=0$ to $l\to\infty$ instead of cutting off at $l_{\max}$. Thus Eqs. (7) and (8) actually describe a truncated δ function and we see that the deformed nucleus must be aligned perpendicular, or very nearly perpendicular, to the beam axis. The function $\delta(\cos\theta-\cos\theta_0)$ can be regarded as an eigenstate of the operator $\cos\theta$ with eigenvalue $\cos\theta_0$. Taking the experimental assertion that the observed resonance corresponds to a 6α-chain state, together with the result that the state has polar angle of 90° and uniformly distributed azimuth, we obtain the picture of a flat disk perpendicular to the beam (since there must be azimuthal symmetry). From the value deduced for $l_{\max}$, and thus $R_{\max}$ (see below), we have some idea of the radius of this disk. Thus the reaction effectively produces an aligned deformation on resonance—a shape eigenstate.

It should be noted that the ideas developed above do not hold for elastic scattering or for strongly coupled inelastic scattering. For elastic scattering $\delta_i=\delta_f$ and so $\exp(i\delta_i)\exp(i\delta_f)$ is then always -1 on resonance. Finally we can make a crude estimate of the length of the chain using the expression $k_f R=[l_{\max}(l_{\max}+1)]^{1/2}$, where k_f is the wave number in the final channel and we set $l_{\max}=18$. This yields $2R\approx 18$ fm for the length of the chain. This large value for R indicates that the reaction may already begin to take place while the nuclei are still well separated, so that the low partial waves are not completely absorbed and their inclusion in the sum of Eq. (7) is quite appropriate. This chain length is somewhat smaller than the cluster model calculations predict (i.e., approximately 24 fm [7]); nevertheless, it does indicate that a very deformed system has been produced.

In summary, the new type of resonant behavior we have described in this Letter should be a very general phenomenon. The necessary conditions for its observation are that there be either an inelastic scattering involving two weakly coupled channels or a rearrangement collision involving distinct entrance and exit channels, only one of which is resonant. This resonant channel should contain several quasibound states (with different angular momentum values, but belonging to a band characterized by a fixed value of $2n+l$) which are effectively degenerate. On resonance, successive even partial wave amplitudes will then have a phase $e^{-i\pi l/2}$ leading to an angular distribution with a strong local maximum at 90° several times larger than any nearby local maximum. The quantal deflection function will reduce to $|\Theta|\approx\pi/2$. Finally, although a system of identical bosons will accentuate this effect since odd partial waves are absent from the scattering, it is not an essential prerequisite for observing the effect.

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