Entropy of a Classical Stochastic Field and Cosmological Perturbations

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We propose a general definition of nonequilibrium entropy of a classical stochastic field. As an example of particular interest in cosmology we apply this definition to compute the entropy of density perturbations in an inflationary Universe. On the scales of structures in the Universe, the entropy of density perturbations dominates over the statistical fluctuations of the entropy of cosmic microwave photons, indicating the relevance of the entropy of density fluctuations for structure formation.

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(1) Introduction.—The concept of entropy contains relevant information about a dynamical system. In systems with a finite number of degrees of freedom there is a natural way to define entropy, even if the system is out of thermal equilibrium. We are interested in systems with infinitely many degrees of freedom and which can be described by a stochastic Gaussian field. In this Letter, we use Shannon's [1] general definition of entropy to calculate the nonequilibrium entropy of a stochastic Gaussian field.

An issue of considerable interest is the development of a consistent definition of entropy in general relativity and cosmology. There have been some key results in this area. The observation that all information about a particle crossing the Schwarzschild horizon is lost led Bekenstein [2] and Hawking [3] to their famous formula for the entropy of a black hole. Penrose [4] suggested that it may be possible to realize the second law of thermodynamics in cosmology by assigning an entropy to the gravitational field itself. He conjectured that the plausible definition of entropy might be "some integral" of the Weyl tensor squared, and that the Universe starts in a state of minimal gravitational entropy. In this picture structure formation and the second law of thermodynamics are reconciled, since gravitational clustering leads to an increase in the Weyl tensor, thus generating gravitational entropy. The entropy of gravitational waves was studied in Ref. [5], and there has also been a lot of work [6] on the entropy of particles produced in strong gravitational fields.

In this Letter we use our formula for the nonequilibrium entropy of a stochastic Gaussian field to propose a new approach to the problem of gravitational entropy in cosmology. The formalism is based on separating the entire system of gravitational plus matter fields into background fields (chosen to have high space-time symmetries) and linearized fluctuating fields; the latter are the stochastic fields to which we apply our general definition of entropy.

It has been demonstrated that the dynamics of perturbations can be reduced to the dynamics of a single scalar field (which comprises in a self-consistent manner both scalar and/or tensor gravitational field perturbations and matter field fluctuations) in the classical space-time background (for a recent review see Ref. [7]). The evolution of the background field is completely specified; this means it carries *no* entropy. On the other hand, the fluctuating field carries significant entropy. This statement needs justification.

In order to obtain growth of entropy, it is necessary to propose some kind of coarse graining in which some information is lost during the evolution. In this work, we consider a free scalar field in an expanding space-time background, in which there is abundant production of perturbations by parametric amplification [8]. We assume that there is a mechanism which generates stochasticity in the phases of the perturbations produced during the evolution. This mechanism is effective for the field modes within the horizon, and generates entropy in the fluctuating field.

The Letter is organized in the following way. The next section is devoted to the derivation of the formula for the entropy of a stochastic Gaussian scalar field. Using this formula, we then calculate (in Sec. 3) the entropy of cosmological density perturbations. In Sec. 4 we hint at some additional possible applications of the formula in the context of cosmology.

(2) Entropy of classical field.—We wish to consider the entropy associated with a classical stochastic field. For a given real scalar field ϕ and its canonical momentum π, there is a probability distribution functional $P[\varphi,\pi]$ defined over an infinite-dimensional space spanned by functions $\{\varphi,\pi\}$. The probabilistic definition of entropy gives

$$S=-\int P[\varphi,\pi]\ln P[\varphi,\pi]\,\mathcal{D}\varphi\,\mathcal{D}\pi\,, \tag{2.1}$$

where the probability functional $P[\varphi,\pi]$ is normalized to unity. We assume a Gaussian process, i.e., that the knowledge of two-point correlation functions suffices to completely specify the stochastic properties of the fields φ and π. (Higher-order correlation functions can be then given in terms of two-point correlations.)

If the stochastic process is non-Gaussian, then the Gaussian approximation may still be a good one, provided corrections due to higher-order correlations are small. An additional requirement is that correlations are of finite range. This is fulfilled in the cosmological setup, because a natural cutoff for correlation is the horizon scale.

The Gaussian approximation breaks down when the perturbations grow nonlinear, and effects of the nonlinearities (originating in the full theory) become significant. In this case the corrections arising in the higher-order correlation functions become important and need to be incorporated in the probability distribution.

The definition of entropy in Eq. (1) can be applied to cosmological perturbations in an expanding universe (see Sec. 3), when the evolution of perturbations can be well represented by a Hamiltonian of second order in φ and π. This means that the Gaussian character of the probability distribution is preserved in the course of evolution.

We now sketch a derivation of the expression for entropy in terms of correlation functions. Assume that at some time t the probability functional $P[\varphi,\pi]$ has a general Gaussian form,

$$P[\varphi,\pi]=\frac{1}{\mathcal{N}}\exp\left[-\int\frac{1}{2}\{\varphi(\mathbf{x})A(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})+\pi(\mathbf{x})B(\mathbf{x},\mathbf{y})\pi(\mathbf{y})+2\varphi(\mathbf{x})C(\mathbf{x},\mathbf{y})\pi(\mathbf{x},\mathbf{y})\}d^3x\,d^3y\right], \tag{2.2}$$

where $\mathcal{N}$ is a normalization constant, and A, B, and C are related to the two-point correlation functions in a way yet to be determined. In a homogeneous space-time background, A, B, and C are functions of $\mathbf{x}-\mathbf{y}$ only.

By a clever substitution, it is possible to bring Eq. (2.2) into a diagonal form, in which φ and π are replaced by new normal coordinates. It is then quite straightforward to evaluate the normalization factor $\mathcal{N}$ of Eq. (2.2),

$$\mathcal{N}=\sqrt{\det\mathcal{D}(\mathbf{x}-\mathbf{y})}\,, \tag{2.3}$$

where $\mathcal{D}$ can be expressed in terms of correlation functions,

$$\mathcal{D}(\mathbf{x}-\mathbf{y})=\int d^3z[\langle\varphi(\mathbf{x})\varphi(\mathbf{z})\rangle\langle\pi(\mathbf{z})\pi(\mathbf{y})\rangle-\langle\varphi(\mathbf{x})\pi(\mathbf{z})\rangle\langle\pi(\mathbf{z})\varphi(\mathbf{y})\rangle]\,. \tag{2.4}$$

Using Eqs. (2.2) through (2.4), the expression (2.1) for the entropy gives

$$S=\mathrm{Tr}\,\delta(\mathbf{x}-\mathbf{y})+\ln\mathcal{N}\,. \tag{2.5}$$

The first term is an irrelevant constant. The relevant contribution comes from the second term and can be rewritten as

$$S=\tfrac{1}{2}\ln\det\mathcal{D}(\mathbf{x}-\mathbf{y})\,. \tag{2.6}$$

The above equation is the main result of this section. Equations (2.4) and (2.6) can be used to obtain the entropy of any stochastic classical scalar field whose probability distribution can be well approximated by the Gaussian probability distribution (2.2).

In order to calculate the determinant of $\mathcal{D}(\mathbf{x},\mathbf{y})$, one needs to solve the eigenvalue problem associated with $\mathcal{D}$. Under quite general conditions (assuming $\mathcal{D}$ is of finite support) and using the ζ-function regularization scheme, it is possible to show [9] that this determinant can be expressed in terms of the spectral density $\mathcal{D}_{\mathbf{k}}$, which is given by the Fourier transform of $\mathcal{D}(\mathbf{x}-\mathbf{y})$,

$$\begin{aligned}\mathcal{D}_{\mathbf{k}}&\equiv\int d^3z\,e^{-i\mathbf{k}\cdot\mathbf{z}}\mathcal{D}(\mathbf{z})\\&=\langle|\varphi_{\mathbf{k}}|^2\rangle\langle|\pi_{\mathbf{k}}|^2\rangle-|\langle\varphi_{\mathbf{k}}\pi_{-\mathbf{k}}\rangle|^2\,,\end{aligned} \tag{2.7}$$

and it is positive definite. The entropy then reads

$$S=V\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\ln\mathcal{D}_{\mathbf{k}}\,. \tag{2.8}$$

The procedure to calculate the entropy of a classical Gaussian field is now very simple. Given two-point correlation functions, one calculates $\mathcal{D}(\mathbf{x}-\mathbf{y})$ [Eq. (2.4)], Fourier transforms it [Eq. (2.7)], and obtains the entropy according to Eq. (2.8). We now apply this prescription to an example which is of interest in cosmology.

(3) Entropy of cosmological perturbations.— In this section we apply the method developed above to calculate the entropy of cosmological density perturbations. This is an example of relevance in cosmology, because it is likely that the scalar density perturbations seed structures in the Universe. We find that the entropy of scalar density perturbations on large scales in the Universe is significant when compared to the statistical fluctuations of the entropy of cosmic microwave photons on the same scales.

Before we present any calculations, we give a short summary of the theory of density perturbations. Density perturbations are scalar-type metric perturbations which couple to energy density and pressure. For matter which is in the form of a scalar field, or an ideal gas, it turns out that density perturbations can be described in a self-consistent manner in terms of a single gauge-invariant scalar field φ, which is a linear combination of scalar field matter fluctuations (ideal gas density fluctuations) and longitudinal metric fluctuations and whose dynamics is given by a quadratic action. (For a comprehensive account of the gauge-invariant formalism of linear cosmological perturbation in Friedmann-Robertson-Walker backgrounds, see Ref. [7]; for a pedagogical introduction, see Ref. [10].)

Hence, assuming that the gauge-invariant field φ is a stochastic Gaussian field, we can calculate the entropy associated with φ using Eqs. (2.4) and (2.6), or equivalently Eqs. (2.7) and (2.8). In order to accomplish this, we need to know the two-point correlation functions $\langle\varphi(\mathbf{x},t)\varphi(\mathbf{y},t)\rangle$, $\langle\pi(\mathbf{x},t)\pi(\mathbf{y},t)\rangle$, and $\langle\varphi(\mathbf{x},t)\pi(\mathbf{y},t)\rangle$, where $\pi(\mathbf{x},t)=\partial\varphi(\mathbf{x},t)/\partial t$.

As an example of a model in which the correlation functions exhibit nontrivial behavior, we consider an expanding universe with initial quantum fluctuations which evolve into classical ones as a result of evolution. In this case, particle pairs are produced via parametric amplification, i.e., via coupling of matter fields to the nontrivial space-time background. Because of the generation of perturbations, the correlation functions become time dependent. Here we consider the inflationary universe scenario in which there is abundant production of inhomogeneities.

The Hamiltonian governing the evolution of the single scalar field φ and momentum π is quadratic in φ and π, so that it is convenient to represent the evolution operator $\hat{U}(t)$ in a form in which the effects of free evolution ($\mathcal{R}$) and interaction with the background (S) are separated. $\hat{U}(t)$ is the product of the rotation operator $\mathcal{R}$ and the two-mode squeeze operator S,

$$\hat{U}(t)=\mathcal{R}(\{\theta_{\mathbf{k}}\})S(\{r_{\mathbf{k}},\varphi_{\mathbf{k}}\}), \tag{3.1}$$

where

$$\mathcal{R}(\{\theta_{\mathbf{k}}\})=\prod_{\mathbf{k},k_x>0}\mathcal{R}(\theta_{\mathbf{k}}),$$
$$S(\{r_{\mathbf{k}},\varphi_{\mathbf{k}}\})=\prod_{\mathbf{k},k_x>0}S(r_{\mathbf{k}},\varphi_{\mathbf{k}}). \tag{3.2}$$

The product $\prod_{\mathbf{k},k_x>0}$ is over half of the possible values of momenta $\mathbf{k}$ (for definiteness, say $k_x>0$). The rotation angles $\theta_{\mathbf{k}}=\int^t\omega_{\mathbf{k}}(t')dt'$ are given in terms of the frequency $\omega_{\mathbf{k}}$ of mode $\mathbf{k}$; $r_{\mathbf{k}},\varphi_{\mathbf{k}}$ are the squeeze factor and phase, respectively, and can be expressed in terms of parameters of the Hamiltonian (see Ref. [11]). The two-mode squeeze operator $S(r_{\mathbf{k}},\varphi_{\mathbf{k}})$ acts on the vacuum $|0_{\rm in}\rangle$, creating pairs of particles with momenta $\mathbf{k}$ and $-\mathbf{k}$, so that the total momentum of the pair is zero; hence $S(r_{\mathbf{k}},\varphi_{\mathbf{k}})$ is a momentum conserving operator. The operator S mathematically describes the process of parametric amplification.

Now we can express the two-point correlation functions of quantum operators $\hat{\varphi}$ and $\hat{\pi}$ in terms of the parameters of the squeezed state which is obtained as a result of the evolution of the initial vacuum state $|\theta_{\rm in}\rangle$ of cosmological perturbations in an expanding universe. Simple, but rather lengthy calculation gives [12]

$$\langle 0_{\rm in}|\varphi(\mathbf{x},t)\varphi(\mathbf{y},t)0_{\rm in}\rangle=\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{2\omega_{\mathbf{k}}(t)}\left[(2\sinh^2r_{\mathbf{k}}+1)-\sinh2r_{\mathbf{k}}\cos2\left(\int\omega_{\mathbf{k}}dt-\varphi_{\mathbf{k}}\right)\right], \tag{3.3}$$

where the frequencies $\omega_{\mathbf{k}}$ and the squeeze factor $r_{\mathbf{k}}$ depend on time because of the nontrivial evolution of the background. Similar expressions are obtained for $\langle 0_{\rm in}|\pi(\mathbf{x},t)\pi(\mathbf{y},t)|0_{\rm in}\rangle$ and $\langle 0_{\rm in}|\varphi(\mathbf{x},t)\pi(\mathbf{y},t)|0_{\rm in}\rangle$.

To associate entropy with the final state, we must neglect some information. Typically, this will be information which is very sensitive to any kind of perturbation, either of the state or of the system. In our example, the phases $2\int\omega_k\,dt-\varphi_k$ will depend sensitively on a perturbation, whereas the amplitudes will not. Hence, we can coarse grain the system, average over the phases, and replace the above correlation functions by reduced correlation functions where the phase-dependent terms vanish. The coarse graining leads to decoherence, which is a necessary condition for the quantum to classical transition [13]. Provided there is decoherence and $\sinh r_k\gg1$, then we can take the classical limit in which classical correlation functions can be identified with quantum expectation values, i.e, $\langle\varphi(x)\varphi(y)\rangle=\langle 0_{\rm in}|\varphi(x)\varphi(y)|0_{\rm in}\rangle$. This issue is discussed in more detail in Ref. [9].

As a result we obtain for the spectral density of the operator $\mathcal{D}(\mathbf{x}-\mathbf{y})$

$$\mathcal{D}_{\mathbf{k}}=\sinh^2r_{\mathbf{k}}(1+\sinh^2r_{\mathbf{k}}). \tag{3.4}$$

The entropy density per mode is then [Eq. (2.6)]

$$s_{\mathbf{k}}=S_{\mathbf{k}}/V=\tfrac{1}{2}\ln\mathcal{D}_{\mathbf{k}} \tag{3.5}$$

and $\mathcal{D}_{\mathbf{k}}\approx n_{\mathbf{k}}^2=\sinh^2r_{\mathbf{k}}$ (for $n_{\mathbf{k}}\gg1$), where $n_{\mathbf{k}}$ is the average number of particles in the $\mathbf{k}$ mode, whenever the notion of particles can be defined.

An advantage of our formula for the entropy is that it is independent of the definition of particles (and hence is not subject to the ambiguities of particle notion in a nontrivial space-time). Equation (3.5) can be applied even if the notion of a particle is not well defined, e.g., for inhomogeneities in a matter-dominated universe [7]. In cases where the notion of a particle is well defined, the entropy of Eq. (3.5) agrees with the usual expression in terms of occupation numbers.

It is useful to define the entropy density s_λ per logarithmic wavelength $\lambda\sim1/k$ interval

$$s_\lambda\sim k^3s_{\mathbf{k}}\approx(1/\lambda^3)\ln\mathcal{D}_{\mathbf{k}}. \tag{3.6}$$

To demonstrate how the technique developed above works, we now apply Eq. (3.5) to estimate the entropy of cosmological perturbations produced during the inflationary stage of a model of chaotic inflation [14]. The simplest potential for the inflation field ϕ_I is $V(\phi_I)=\frac{1}{2}m^2\phi_I^2$, where m is the mass of the inflation, typically of the order 10^{13} GeV. Considering perturbations on scales which enter the horizon late in the radiation era, we obtain the following result for the entropy density of perturbations on a typical scale $\lambda_{\rm ph}\sim a/k$ (see Ref. [9]):

$$s_{\lambda_{\rm ph}}\sim\frac{1}{\lambda_{\rm ph}^3}\ln\left[(ml)\ln\frac{\lambda_{\rm ph}}{\lambda_\gamma}\left(\frac{\lambda_{\rm ph}}{t}\right)\left(\frac{\lambda_{\rm ph}}{l}\right)\right], \tag{3.7}$$

where l is the Planck length, t is the cosmological time, and λ_γ is the typical wavelength of the cosmic microwave background radiation. This formula is applicable for perturbations which satisfy the condition $t \gg \lambda_{ph} \gg \lambda_\gamma$. The contribution to the total entropy of galactic scale perturbations ($\sim$1-100 Mpc) per corresponding galactic volume is $S_{gal} \sim \lambda_{gal}^3 s_{gal} \approx 200-220$. The entropy of gravitational radiation can be estimated in the same manner (see Ref. [9]).

Statistical fluctuations of cosmic microwave photons are another potential source of inhomogeneities. However, the entropy density of these fluctuations scales as $\lambda^{-3/2}$ compared to the logarithmic dependence we found for the entropy density of density perturbations. Hence, on scales of galaxies, the entropy of gravitational perturbations dominates over the statistical fluctuations of the entropy of cosmic microwave photons. The total entropy of cosmological fluctuations is, however, suppressed by a factor $(H/m_{Pl})^{3/2}$ (where H is the Hubble expansion rate at the end of inflation) compared to that of the cosmic microwave background. The dominance of the entropy of gravitational perturbations on large scales is a sign of the relevance of this entropy for structure formation. A further application [9] of this entropy is in the context of a collapsing universe.

It is worth noting that because of the weak (logarithmic) dependence of entropy per mode on the energy scale m of the model, our conclusion remains valid for a wide class of inflationary models.

(4) Discussion.— We derived a new formula for the entropy of a stochastic Gaussian scalar field in terms of two-point correlation functions. We then applied this result to the cosmologically relevant example of density perturbations, using the formalism of linearized gauge-invariant scalar perturbations about a homogeneous classical space-time and matter background. We found that the entropy of the system (scalar gravitational metric perturbations plus matter density fluctuations) grows as a logarithm of the number of particles created as the Universe expands. On the scales of large structures in the Universe, the entropy of density perturbations in an inflationary universe dominates over the entropy of statistical fluctuations of the cosmic microwave photons.

The formalism for calculation of entropy which we developed in this Letter can be applied to any (cosmological) problem, which can be reduced to the evolution of a classical stochastic Gaussian field.

The treatment of the full (nonlinear) gravitational field is still an open problem, and the corresponding formula for entropy is yet to be constructed. It would be very instructive to demonstrate that the entropy of the gravitational field continues to grow, even when the perturbations become nonlinear.

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[1] C. Shannon, Bell. Syst. Tech. J. **27**, 379 (1948); 27, 623 (1948); E. Jaynes, Phys. Rev. **106**, 620 (1957).
[2] J. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[3] S. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[4] R. Penrose, in *General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979).
[5] L. Smolin, Gen. Relativ. Gravitation **17**, 417 (1985).
[6] B. Hu, Phys. Lett. **123B**, 189 (1983); B. Hu and D. Pavon, Phys. Lett. B **180**, 329 (1986); H. Kandrup, Classical Quantum Gravity **3**, L55 (1986); J. Math. Phys. **28**, 1398 (1987); B. Hu and H. Kandrup, Phys. Rev. D **35**, 1776 (1987); H. Kandrup, Phys. Rev. D **37**, 3505 (1988); E. Calzetta and B. Hu, Phys. Rev. D **37**, 2878 (1988); S. Habib and H. Kandrup, Ann. Phys. (N.Y.) **191**, 335 (1989).
[7] V. Mukhanov, H. Feldman, and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[8] L. Parker, Phys. Rev. Lett. **21**, 562 (1968); Phys. Rev. **183**, 1057 (1969); L. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 (1974)].
[9] R. H. Brandenberger, V. Mukhanov, and T. Prokopec, Brown University Report No. BROWN-HET-849, 1992 (to be published).
[10] R. Brandenberger, H. Feldman, and V. Mukhanov, in *Proceedings of the International Conference on Gravitation and Cosmology, 1991* (Wiley Eastern Ltd., New Delhi, 1992) (Brown University Report No. BROWN-HET-841, 1992).
[11] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); **31**, 3093 (1985).
[12] T. Prokopec, Brown University Report No. BROWN-HET-861, 1992 (to be published).
[13] See, e.g., W. Zurek, Phys. Rev. D **24**, 1516 (1981); E. Joos and H. Zeh, Z. Phys. B **59**, 223 (1985); M. Gell-Mann and J. Hartle, J. Phys. A **24**, L159 (1991); J. Paz and S. Sinha, Phys. Rev. D **45**, 2823 (1992).
[14] A. Linde, Phys. Lett. **129B**, 177 (1983).