Algebraic Correlations in Conserving Chaotic Systems

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It is argued that for $d > 1$, chaotic systems with a conserved quantity can exhibit generic scale invariance—algebraic decay of spatial correlations without the tuning of external parameters—with exponents analytically calculable from noisy nonchaotic models. Numerical confirmation of these predictions for a specific coupled map system is presented.

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Over the past few years it has become clear that a broad class of nonequilibrium systems with external white noise exhibits generic scale invariance—correlations that decay like power laws in both space and time without the tuning of external parameters. The first example of this phenomenon was provided by fluids in a temperature gradient, where the static structure factor was argued theoretically [1] to diverge for arbitrary parameter values, a prediction recently verified experimentally [2]. It has since been shown analytically [3–5], and verified in part numerically [6], that, at least in one-component systems, local conservation is a necessary [7] and almost sufficient [8] condition for generic algebraic correlations in noisy driven systems. One-dimensional (1D) systems, wherein local conservation laws are typically not sufficient to produce scale invariance, are the main exception. For systems with a single-component field on hypercubic lattices in $d > 1$ space dimensions, spatial correlations are predicted [4,5] to decay as $1/r^d$ and $1/r^{d+2}$ in systems that respectively break or respect the hypercubic symmetry.

In deriving these results from perturbative renormalization-group (RG) arguments [4], one tacitly assumes that the systems in question have modest nonlinearities and so are nonchaotic. It remains to be seen whether generic scale invariance continues to hold in noisy conserving systems driven strongly enough to produce chaos. One is further led to ask whether in *noiseless* conserving chaotic systems the chaotic fluctuations simulate [9] the effect of external noise and so produce algebraic correlations. In this paper we propose heuristic arguments to suggest that both in the presence and in the absence of noise, many conserving chaotic systems do indeed exhibit generic scale invariance for $d > 1$. We present numerical evidence for coupled map lattices with $d = 2$ and $d = 1$ which supports these conclusions, and further indicates, in agreement with the heuristic arguments, that the exponents characterizing the power-law decays in chaotic systems are the same as those predicted analytically for noisy nonchaotic ones. These results suggest that in the presence of local conservation generic scale invariance with simple predictable exponents may hold as widely in chaotic systems, whose analytic intractability is notorious, as in noisy nonchaotic ones.

We now discuss the reasons to expect generic scale invariance in conserving chaotic systems. First consider systems with external white noise. In such systems the onset of chaos, as a control parameter is varied, is often a purely local phenomenon, rather than a collective one or phase transition: One can regard the chaotic fluctuations as simply superimposed upon the underlying nonchaotic phase, the correlations between the fluctuations at different sites being purely short ranged. (Arguments and supporting numerical evidence for this assertion are given in Refs. [10] and [11].) Since there is no change of phase, the generic algebraic decays of spatial and temporal correlations characterizing the nonchaotic regime continue to hold in the chaotic one, and with the same exponents. Next imagine reducing the external noise to zero in the chaotic state (assuming, without loss of generality, that the chaos persists in the noiseless limit). Though one cannot argue with certainty that the asymptotic correlations are unaltered in this limit, given that the effect of noise is typically decorrelating, one's strong expectation is that correlations in noiseless systems should decay at least as slowly as those in noisy ones. One concludes that conserving, noiseless, chaotic systems can also display generic scale invariance, with exponents which, if they differ at all from the noisy case, correspond to *slower* decays.

To test these ideas we study coupled conserving maps [12] evolving in discrete time on square lattices with periodic boundary conditions. Let $S_t(i)$ denote the dynamical variable at site i at time t. The updating is synchronous and is given by

$$S_{t+1}(i)=S_t(i)+\frac{\nu}{4}\sum_j [f(S_t(j))-f(S_t(i))] + \frac{\alpha}{2}[S_t^2(i)-S_t^2(i+2\hat{\mathbf{x}})]+\eta_t(i)\,, \qquad (1)$$

where j denotes the four nearest neighbors of i, $\hat{\mathbf{x}}$ is a unit vector along the x axis, and $f(S)=S-S^3$. The control parameter ν regulates the nonlinear diffusive coupling between sites and the third term introduces x-y anisotropy and breaks reflection symmetry in x, as does the last term, $\eta_t(i)$, a noise variable generated from a second set of noise variables $\tilde{\eta}_t(i)$ so as to satisfy the conservation law $\eta_t(i)=\tilde{\eta}_t(i+\hat{\mathbf{x}})-\tilde{\eta}_t(i-\hat{\mathbf{x}})$. The $\tilde{\eta}_t(i)$ are chosen independently and randomly from either a Gaussian distribution with $\langle\tilde{\eta}_t(i)\rangle=0$ and $\langle\tilde{\eta}_t(i)\tilde{\eta}_{t'}(j)\rangle=\sigma^2\delta_{i,j}\delta_{t,t'}$, or (as is more typical in the simulations) from a distribution function uniform for $|\tilde{\eta}|\leq\sigma$ and 0 otherwise. Finally, the rule (1) conserves the variable $S_t(i)$ locally: $\rho=N^{-1}\sum_{i=1}^{N}S_t(i)$ is independent of t, where N is the number of sites.

Let us now briefly summarize part of the phase diagram of model (1). For small ν and in the presence of small noise the system is in a spatially homogeneous 1-cycle phase. Straightforward linear stability analysis shows that at $\nu=\nu_1^*=1/(1-3\rho^2)$ this phase first becomes unstable at the wave vector $\mathbf{q}=\mathbf{Q}\equiv(\pi,\pi)$. The system undergoes a transition into a spatially ordered, checkerboard ("antiferromagnetic"), temporal 2-cycle phase in which the odd and even sublattices assume values a and b at alternate times. We refer to this as the AF I phase. As ν is increased beyond [13] $\nu_2^*=\frac{3}{2}\nu_1^*$, the system undergoes a transition into a temporal 2-cycle wherein the odd sites assume values a_1 and a_2 at odd and even times, respectively, while the even sites assume values b_1 and b_2 consistent with the conserved value of ρ. Thus the antiferromagnetic order parameter develops a nonzero temporal average. We refer to this as the AF II state. Upon further increase of ν the model displays a chaotic two-band phase with long-ranged antiferromagnetic order, i.e., local, chaotic fluctuations superimposed on the AF I state. The power spectrum has a characteristic broadband background, with sharp features at frequencies $\omega=\pm\pi$ corresponding to the two-band oscillations. We take positive Lyapunov exponents as the signature of chaotic phases.

For zero noise other phases (e.g., temporal 4-cycles) can occur, and the phase diagram can, not surprisingly, depend on initial conditions. As our goal is to study correlations in chaotic phases rather than details of phase diagrams, we work exclusively in the chaotic two-band phase, as far as possible from transitions to other phases. Though for sufficiently large ν, model (1) can become unstable, and runaways to arbitrarily large values of $S(i)$ can occur, we consider only ν's for which, at least within our numerical limitation of several million time steps per site, the variables remain bounded. One can argue analytically that there exists, for zero noise, a range of ν above the onset of chaos for which the variables remain bounded by unity provided their initial values are bounded.

The results are given in Figs. 1–4. Each of these figures shows a log-log plot (note that natural logarithms are used in this paper) of the absolute value [14] of the equal-time spatial correlation function $G(\mathbf{r})=\langle[S_t(i)-\langle S_t(i)\rangle][S_t(j)-\langle S_t(j)\rangle]\rangle$ vs r, for $\mathbf{r}$ along the x axis: $\mathbf{r}=r\hat{\mathbf{x}}$; similar results hold for $\mathbf{r}=r\hat{\mathbf{y}}$. The angular brackets represent averages over time and over $\mathbf{i}$ and $\mathbf{j}$ with $\mathbf{r}=\mathbf{i}-\mathbf{j}$; data for values of r up to 40 or 50 are shown. The data displayed were taken for $\rho=0.10$, but the results are independent of ρ.

Figure 1 shows, as a control, data for the nonchaotic, temporal 1-cycle, spatially uniform phase in the presence of noise. The straight line in the figure has a slope of -2, the theoretical prediction for noisy anisotropic phases in two dimensions [4,5]; the data are nicely consistent with this value.

Figure 2 shows data for the (nonchaotic) AF I phase in the presence of anisotropic noise. (Similar results were obtained for the AF II phase.) The slope is again consistent with the value of -2. Note that although the -2 result was derived in Ref. [4] for spatially uniform phases, essentially identical arguments can be made for the ordered checkerboard phases that occur here. Note that the data for odd and even values of r in Fig. 2 are staggered. Indeed, these data are consistent with algebraic decay both in the $q=0$ and $q=Q$ modes. This seems surprising at first glance, since the antiferromagnetic order parameter is not conserved in the system. the result can, however, be understood by considering a simple Langevin model with two coupled variables, one con-

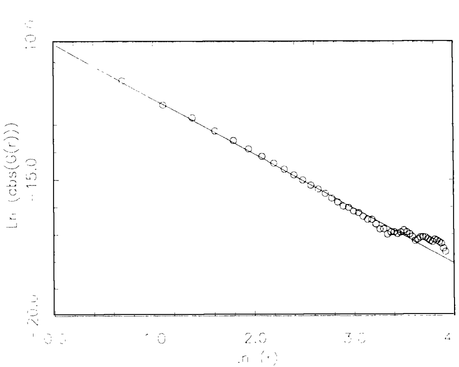

FIG. 1. Log-log plot (natural logarithms) of $G(r\hat{\mathbf{x}})$ vs r for the (nonchaotic) 1-cycle phase of model with external noise (1), with $\nu=0.8$, $\alpha=0.25$, $\rho=0.10$, and $\sigma=0.02$, on a 100×100 lattice. The straight line has slope -2 consistent with previous theoretical expectations.

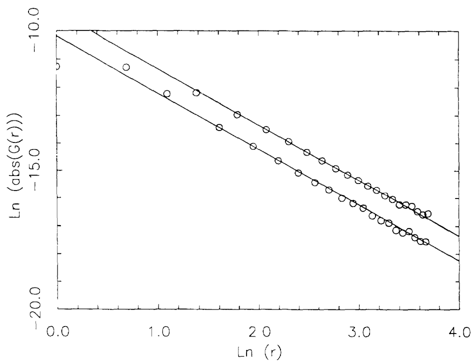

FIG. 2. Log-log plot of $G(r\hat{\mathbf{x}})$ vs r for AF I phase of model with external noise (1), with $\nu=1.30$, $\alpha=0.25$, $\rho=0.10$, and $\sigma=0.02$, on an 80×80 lattice. The straight lines have slope -2. Note the staggering of points for even and odd r showing "induced scale invariance" in the $\mathbf{Q}=(\pi,\pi)$ mode.

served and the other not. It can be shown [15] that the conserved mode induces power-law spatial correlations in the nonconserved one. Thus our numerical data show this phenomenon of "induced scale invariance."

Figure 3 shows data for the chaotic checkerboard (AF I) phase in the presence of external noise. The maximum Lyapunov exponent was calculated to be 0.415 ± 0.01. Again, the straight line indicating scale invariance has slope -2, consistent with the heuristic prediction that the asymptotic behaviors of chaotic and nonchaotic phases with external noise are identical.

In Fig. 4 we show data for $G(r\hat{\mathbf{y}})$ in the *chaotic* checkerboard (AF I) phase with *no* noise, for parameters for which the maximum Lyapunov exponent is approximately 0.42. The results are in good agreement with the heuristic arguments presented earlier, yielding an exponent of -2. Similar results hold along the x direction. It is important to emphasize that no parameters have been tuned to produce the observed power-law decays. We have tak-

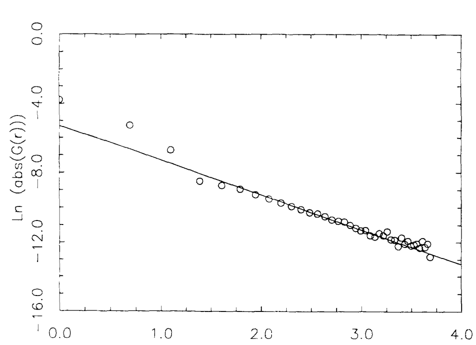

FIG. 3. Log-log plot of $G(r\hat{\mathbf{x}})$ vs r for the *chaotic*, checkerboard (AF I) phase of model (1) with noise, with $\nu=1.91$, $\alpha=0.00$, $\rho=0.10$, and $\sigma=0.05$, on an 80×80 lattice. The straight line has slope -2.

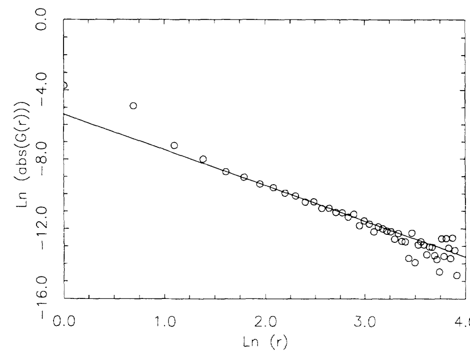

FIG. 4. Log-log plot of $G(r\hat{\mathbf{y}})$ vs r for the *chaotic*, checkerboard (AF I) phase of the *deterministic* version of the model (1), with $\nu=1.91$, $\alpha=0.25$, $\rho=0.10$, and $\sigma=0.00$, on a 100×100 lattice. The straight line has slope -2.

en data for several values of ν in the chaotic checkerboard phase, to verify that the results are indeed characteristic of the entire phase. Thus the system with only deterministic chaos displays power laws consistent with the predictions for a noisy laminar state.

As another check, we have computed $G(r)$ in a 1D version of model (1), without noise, in the chaotic phase. The data do not fit a straight line on a log-log plot, consistent with the theoretical prediction for noisy, nonchaotic systems that generic scale invariance does not occur in 1D. The structure factor typically shows a broad peak at some q near Q indicating short-ranged order. Figure 5 shows the data on a semilogarithmic plot; the data are consistent with a correlation length of 2 to 3 lattice spac-

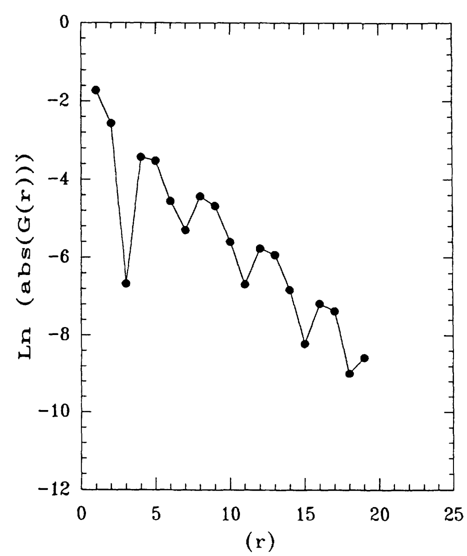

FIG. 5. Log-linear plot of $G(r)$ vs r for the chaotic phase of the 1D version of model (1) without external noise, with $\nu=1.94$, $\alpha=0.25$, $\rho=0.00$, and $\sigma=0.00$, on a 1024 lattice. The data indicate exponential decay (with a correlation length of 2 or 3 lattice spacings) consistent with theoretical expectations.

ings.

These numerical results strongly support both the simple physical arguments predicting generic scale invariance in conserving chaotic systems with $d>1$, and the equivalence of the exponents produced by external noise and deterministic chaotic fluctuations. We have also studied several different conserving coupled map models that break the lattice symmetry other than the one reported here. While it seems plausible to expect that our arguments hold in all these models, in most of them correlations either decayed too rapidly or behaved too irregularly in space to permit any firm conclusion about possible asymptotic power laws to be drawn. In addition, we investigated $G(r)$ in two-dimensional models that respect the full lattice symmetry; however, the expected $1/r^4$ decay was too rapid to obtain persuasive numerical support for the prediction.

Note that the *local* nature of the chaos which seems to underlie the similarity of the large-distance properties of chaotic and nonchaotic phases has a simple interpretation in terms of the RG: The fixed point governing asymptotic properties of the system is independent of the presence of chaos; i.e., the system looks less and less chaotic as one looks at it on progressively longer length scales. (This is the absence of "collective chaos" discussed in Refs. [10] and [11].) To test this idea directly, we studied numerically the histogram of values of the Fourier amplitude $\tilde{S}(\mathbf{Q}=(\pi,\pi),t)$ in the chaotic checkerboard phase. We found that, with or without noise, the width of the histogram decreased like L^{-1} with increasing system size L, consistent with the correlation of chaotic fluctuations being short ranged and so disappearing when averaged over long length scales.

Standard perturbative RG analysis of the continuum version of model (1) with external noise [4] shows that, at least for small coupling constants, the nonlinearities are all irrelevant, i.e., do not alter the long-distance algebraic correlations predicted by the linear theory. Though such analysis need not necessarily provide any information about the strong-coupling problem where chaos appears, our results demonstrate that in fact it does: the noisy linear theory seem to correctly describe the long-distance behavior of the noisy, and even of the noiseless, chaotic phase [16]. If, as we suspect, this holds rather generally [9], then it constitutes a powerful tool for analyzing the macroscopic properties of conserving chaotic systems.

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[7] We reserve the term generic scale invariance for systems with generic algebraic decays of *spatial* correlations. Algebraic decay of correlations in time is much easier to achieve, occurring generically, e.g., even in equilibrium systems with conservation laws. Spatial scale invariance typically implies temporal scale invariance. Note that special symmetries (e.g., the translational invariance of interface models) can produce scale invariance without conservation laws even in equilibrium systems.

[8] Certain nonequilibrium systems, such as sandpiles, which are driven slowly compared with their relaxation rates, are also capable of generic scale invariance, or "self-organized criticality." See P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

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[13] For $\alpha\neq0$ the transition region exhibits complicated behavior. The details are irrelevant for the issues discussed here.

[14] We plot $|G(\mathbf{r})|$ since $G(\mathbf{r})$ need not always be positive.

[15] G. Grinstein, C. Jayaprakash, and J. E. S. Socolar (unpublished).

[16] As a further check we computed the static structure factor $S(\mathbf{q})$ [the Fourier transform of $G(\mathbf{r})$] for small wave numbers $\mathbf{q}$. For the noisy nonchaotic phases of model (1), $S(\mathbf{q})$ is predicted (Ref. [4]) to behave like $q_x^2/(q_x^2+Aq_y^2)$, where $A\neq1$ is a positive constant. We found, consistent with this prediction, that for the chaotic phase with no noise, $S(\mathbf{q})$ approaches three different constants, the first of them 0, as $\mathbf{q}$ approaches $\mathbf{0}$ along the lines $q_x=0$, $q_y=0$, and $q_x=q_y$, respectively.