Controlling Chaos in High Dimensional Systems

Ditza Auerbach,(1) Celso Grebogi,(1),(2) Edward Ott,(1),(3) and James A. Yorke(2)
(1)Laboratory for Plasma Research, University of Maryland, College Park, Maryland 20742
(2)Institute for Physical Sciences and Technology and Department of Mathematics, University of Maryland, College Park, Maryland 20742
(3)Department of Physics and Department of Electrical Engineering, University of Maryland, College Park, Maryland 20742
(Received 29 June 1992)

Recently formulated techniques for controlling chaotic dynamics face a fundamental problem when the system is high dimensional, and this problem is present even when the chaotic attractor is low dimensional. Here we introduce a procedure for controlling a chaotic time signal of an arbitrarily high dimensional system, without assuming any knowledge of the underlying dynamical equations. Specifically, we formulate a feedback control that requires modeling the local dynamics of only a single or a few of the possibly infinite number of phase-space variables.

PACS numbers: 05.45.+b

Feedback controls involving only minute parameter perturbations in an accessible system parameter have recently received much attention as a means of forcing a chaotic system to evolve in a prescribed way [1-5]. The proposed techniques [3,4] suffer from a fundamental limitation when applied to high dimensional physical systems evolving chaotically on a low dimensional attractor. In this Letter, we introduce a feedback control mechanism for controlling such a high, possibly infinite, dimensional system directly from time series data. We stress that neither knowledge of the underlying equations of motion nor a model for them is required.

For a chaotic system whose parameter values are held fixed, one often observes high dimensional transients that vanish with time as the dynamics settle down to a lower dimensional chaotic attractor. The application of even a minute short-term fluctuation in the parameters can reintroduce the higher dimensional transients, rendering it inappropriate to view the dynamics as low dimensional if parameters are frequently tuned. Consequently, the most obvious approach for modeling a system in which an accessible parameter is to be varied is to follow the evolution in the entire set of phase-space variables. Imagine a discrete time n-dimensional system at the nominal parameter value $p=p_0$, possessing an unstable equilibrium state ξ_f that we wish to stabilize. During the application of the control, the trajectory is forced to remain in the vicinity of ξ_f, and the local dynamics can be modeled through the linear relationship

$$\xi_{i+1}-\xi_f=\mathbf{J}(\xi_i-\xi_f)+\mathbf{A}\delta p_i \,, \qquad (1)$$

where ξ_i is the state of the system in its full phase space at time i, $\delta p_i \equiv p_i - p_0$ is the value of the parameter perturbation applied at time i, $\mathbf{J}$ is the local $n\times n$ Jacobian matrix of the underlying map $\xi_{i+1}=\mathbf{F}(\xi_i,p)$ evaluated at ξ_f and $p=p_0$, and $\mathbf{A}=\partial\mathbf{F}/\partial p|_{\xi=\xi_f},\ p=p_0$. The actual perturbation δp_i that should be applied is determined by a feedback rule which depends linearly on the state ξ_i. This approach may not be feasible for large n, since it requires the knowledge of all the components of ξ_i, $\mathbf{J}$, and $\mathbf{A}$. The purpose of this Letter is to describe a simpler approach in which a feedback control can be deduced by modeling the dynamics of only a small number of coordinates.

A chaotic attractor typically has embedded in it a set of unstable periodic orbits [6,7]. A versatile feedback control mechanism which involves alternatively choosing any one of these unstable cycles and then forcing the dynamics to follow the chosen orbit through the application of small external perturbations has been proposed [3], and successfully implemented in an experimental setting [1]. In the feedback control of Ref. [3], the instantaneous phase-space position of the system is identified, and a model of the form of Eq. (1) is deduced locally, in the vicinity of the periodic orbit to be stabilized. For cases in which a chaotic attractor is reconstructed using a time-delay embedding [8,9], a modification to the form of the control law of Ref. [3] has been introduced [4] [the modified law arises due to the presence of an additional term linear in δp_{i-1} in Eq. (1), if ξ is a time-delay vector]. Direct application of these methods to high dimensional systems is problematic due to the necessity of determining an overwhelming amount of information from the data [10]. The novel aspect of the control mechanism we introduce here is that it can be implemented directly from time series data, *irrespective of the overall dimension of the phase space.*

We first develop the control law for the case of an n-dimensional system evolving chaotically on an attractor which has embedded in it, at the nominal parameter value $p=p_0$, an unstable fixed point ξ_f with a one-dimensional unstable manifold of strength λ_1 (in fact, we assume only $|\lambda_1|\geq 1$). At the end of the paper, we will indicate how the control procedure is generalized to unstable orbits of period greater than 1, and orbits possessing more than one unstable direction. Consider measuring a scalar time series $\{x_i\}_{i=1}^{N}$. If at time i the system is in the neighborhood of the fixed point ξ_f, then its evolution is well approximated by the linearization, Eq. (1), whose projection onto the scalar measurement direction is $x_{i+1}=\mathbf{P}_x\mathbf{J}(\xi_i-\xi_f)+\mathbf{P}_x\mathbf{A}\delta p_i$, where $\mathbf{P}_x$ is the projection operator of an n-dimensional vector onto the x direction

($\mathbf{P}_x$ is an n-component row vector). By decomposing the vector $\xi_i - \xi_f$ as a sum of the two vectors $\xi_i^{(s)}$ and $\xi_i^{(u)}$ lying respectively in the stable manifold (contracting directions) and unstable manifold (expanding direction) of $\mathbf{J}$, the projection of Eq. (1) onto the x direction can be expressed as

$$x_{i+1} = \lambda_1 x_i + A_x^{(1)} \delta p_i + \mathbf{P}_x (\mathbf{J} - \lambda_1 \mathbf{I}) \xi_i^{(s)} , \qquad (2)$$

where $A_x^{(1)} = \mathbf{P}_x \mathbf{A}$, and $\mathbf{I}$ is the identity operator. Decomposing $\mathbf{A} = \mathbf{A}^{(s)} + \mathbf{A}^{(u)}$ into vectors lying along the contracting and expanding directions of $\mathbf{J}$, the component of Eq. (1) along the stable manifold of $\mathbf{J}$ has the form $\xi_{i+1}^{(s)} = \mathbf{J}\xi_i^{(s)} + \mathbf{A}^{(s)}\delta p_i$. Using this relation, Eq. (2) can be iterated to obtain an expression for x_{i+k} ($k > 1$) which is linear in the previous k parameter perturbations,

$$x_{i+k} = \lambda_1 x_{i+k-1} + A_x^{(1)} \delta p_{i+k-1} + A_x^{(2)} \delta p_{i+k-2} + \cdots + A_x^{(k)} \delta p_i + \mathcal{R}_x^{(k)} \xi_i^{(s)} , \qquad (3)$$

where $A_x^{(l)} = \mathbf{P}_x (\mathbf{J} - \lambda_1 \mathbf{I}) \mathbf{J}^{l-2} \mathbf{A}^{(s)}$ for $2 \le l \le k$, and $\mathcal{R}_x^{(k)} = \mathbf{P}_x (\mathbf{J} - \lambda_1 \mathbf{I}) \mathbf{J}^{k-1}$. Notice that the scalar form of Eq. (3) is an exact specification to linear order of the local dynamics in the vicinity of ξ_f along the measurement direction, in the presence of parameter perturbations. In contrast with the full phase-space model of Eq. (1), a dependence on the entire history of the parameter perturbations is introduced in the expansion above. The fact that the coefficients $A_x^{(l)}$ decrease exponentially for large l enables control of the dynamics, by utilizing only the leading k coefficients that will be extracted from time series data.

The expansion of Eq. (3) holds for an arbitrary measurement x, but it takes on a particularly simple form if the measurement is made along the unstable direction; that is, along the direction for which $\mathbf{P}_x \mathbf{v} = 0$ for all vectors $\mathbf{v}$ lying in the stable manifold of $\mathbf{J}$. For this special case, the expansion of Eq. (3) reduces to the relation $x_{i+1} = \lambda_1 x_i + A_x^{(1)} \delta p_i$ (for notational convenience, the time coordinate has been translated by $k-1$ units). In the typical case where the measurement x has a component in the stable manifold, the coordinate along the unstable direction, u, can be obtained by subtracting from x components on the stable manifold,

$$u_i = x_i - (\mathbf{P}_x \mathbf{A}^{(s)} \delta p_{i-1} + \mathbf{P}_x \mathbf{J} \mathbf{A}^{(s)} \delta p_{i-2} + \cdots + \mathbf{P}_x \mathbf{J}^{k-2} \mathbf{A}^{(s)} \delta p_{i-k+1}) - \mathbf{P}_x \mathbf{J}^{k-1} \xi_{i-k+1}^{(s)} . \qquad (4)$$

Rewriting Eq. (3) in terms of the variable u, the dynamics takes on the form $u_{i+1} = \lambda_1 u_i + \mathbf{P}_x \mathbf{A}^{(u)} \delta p_i$, characteristic of a one-dimensional invariant set. The unstable fixed point can then be controlled by the feedback rule

$$\delta p_i = -(\lambda_1 / \mathbf{P}_x \mathbf{A}^{(u)}) u_i , \qquad (5)$$

provided $\mathbf{P}_x \mathbf{A}^{(u)} \neq 0$. A single parameter perturbation applied according to this exact control law will displace the trajectory in one time step onto the stable manifold, $u = 0$, of the fixed point ξ_f. Once on this manifold, the trajectory will approach the fixed point at an exponential rate, typically given by λ_2, the contracting eigenvalue of $\mathbf{J}$ of largest magnitude. Thus, knowledge of the unstable coordinate u_i enables the fixed point to be stabilized by the application of a single parameter adjustment lasting only one time step. Subsequent approach to the fixed point is solely governed by the stable manifold of the uncontrolled dynamics.

In practice, the control law of Eq. (5) cannot in general be applied exactly, since it may be impossible to determine the exact values of the coordinate u_i and the projection $\mathbf{P}_x \mathbf{A}^{(u)}$ from measured data. Therefore, we introduce an approximation to the above feedback rule which is based only on knowledge of λ_1 and the first k_0 coefficients $A_x^{(j)}$ appearing in the expansion of Eq. (3) (below we describe how to extract these quantities from a measured time series). From the definition of the $A_x^{(j)}$'s, the constant coefficients appearing in the control law (5) can be expressed as

$$\mathbf{P}_x \mathbf{A}^{(u)} = \sum_{j=1}^{k_0} \frac{A_x^{(j)}}{\lambda_1^{j-1}} + \lambda_1^{-(k_0-1)} \mathbf{P}_x \mathbf{J}^{k_0-1} \mathbf{A}^{(s)} ,$$
$$\mathbf{P}_x \mathbf{J}^l \mathbf{A}^{(s)} = \sum_{j=1}^{l+1} \lambda_1^{l+1-j} A_x^{(j)} - \lambda_1^l \mathbf{P}_x \mathbf{A}^{(u)} \qquad (6)$$

for $l < k_0$. The k_0^{th} approximation to the control law (5) is obtained as follows: The last term in expression (6) for $\mathbf{P}_x \mathbf{A}^{(u)}$ is omitted and in addition the expression for u_i in Eq. (4) is truncated after $k_0 + 1$ terms. The magnitude of the omitted terms decreases exponentially with k_0, yielding an approximate feedback rule whose prescribed gains (the coefficients of $x_i, \delta p_{i-1}, \ldots, \delta p_{i-k_0+1}$) converge to the gains of the exact control law of Eq. (5), exponentially fast in k_0. One expects the convergence of the controlled trajectory to the fixed point ξ_f to be reduced relative to the exact control of Eq. (5). Indeed, for large k_0, the convergence is still exponential with rate $|\lambda_2| + O(\ln k_0 / k_0)$ (see [11] for details). Thus, even though the control procedure takes only a single direction into account explicitly, the actual phase-space trajectory in the full n-dimensional space approaches ξ_f asymptotically. The multi-dimensional control problem is effectively reduced to a one-dimensional one.

The minimum value of k_0 that is sufficient for control depends on the strength of the contraction along the stable manifold of $\mathbf{J}$. In cases where the stable directions of the matrix $\mathbf{J}$ in Eq. (1) are very strongly contracting (the experiment of Ref. [1] exhibits this behavior), a $k_0 = 1$ approximation may be sufficient for controlling the system. We emphasize that, even in this case, the consideration of the high dimensional dynamics is still essential in the proper evaluation of $A_x^{(1)}$.

We now describe how to implement the control procedure for an unstable fixed point of a d-dimensional chaotic attractor. First, a scalar time series output from an experiment at a fixed parameter value $p = p_0$ is embedded in a D-dimensional space ($D > d$ is sufficient [9]), and used to extract the periodic orbit along with its unstable eigenvalue according to the method of Ref. [7]. Next, the $A_x^{(j)}$'s must be obtained experimentally; a trajectory at the nominal parameter value p_0 is generated, and one waits until it enters a small neighborhood of the fixed point ξ_f; i.e., a delay vector in the measurement variable x embedded in $D > d$ dimensions is close to the origin. Such a trajectory remains in the vicinity (the linear regime) of the fixed point, for D time steps, and at each successive iterate j the magnitude of $\xi_j^{(s)}$ decreases relative to $\xi_j^{(u)}$. For a trajectory which approaches the fixed point sufficiently closely, the condition $|\xi_i^{(s)}| \ll |\xi_i^{(u)}|$ will hold at time i, which is several time steps after the initial approach to ξ_f. At time i, the parameter is perturbed by an arbitrary amount δp_i, smaller in magnitude than a prescribed value $\delta p_{\max}$ (small enough so that nonlinearities can be ignored) and switched back to its nominal value p_0 after a single iteration. In the presence of noise, one would repeat this procedure with several close approaches, each time making a different parameter perturbation δp_i, and then determine the least-squares solution $A_x^{(1)}$ to a set of equations each having the form of Eq. (2). The remainder term $\mathcal{R}_x^{(1)}\xi_i^{(s)}$ appearing in this equation can be omitted from the fit since $|\xi_i^{(s)}| \ll |\xi_i^{(u)}|$. Using the value of λ_1 previously determined at $p=p_0$, and the measurements of x, the value of $A_x^{(1)}$ can be determined directly from Eq. (2).

The coefficients $A_x^{(k)}$, for $k>1$ but not too large, can be determined by finding the least-squares solution to a set of linear equations of the form $x_{i+k}=\lambda_1 x_{i+k-1}+A_x^{(k)}\delta p_i$, where x_{i+k} is the measurement k time steps after the parameter perturbation was switched on. We note, however, that in this method nonlinear terms can spoil the determination of $A_x^{(k)}$ if k is too large, since in k time steps the uncontrolled trajectory may escape the neighborhood of ξ_f in which a linear approximation is valid. In order to determine $A_x^{(k)}$ for large k, we utilize all the $A_x^{(j)}$'s with $j<k$ determined previously, in order to hold the trajectory within the linear regime of ξ_f for k iterations. This is accomplished by choosing the parameter perturbations $\delta p_{i+1},\ldots,\delta p_{i+k-1}$ according to the $(k-1)$-order approximation to the control law of Eq. (5), and then determining $A_x^{(k)}$ through the least-squares solution of a set of linear equations, each having the form of Eq. (3) (neglecting the term involving $\mathcal{R}_x^{(k)}$). In general, estimation of the $A_x^{(j)}$'s can be refined by utilizing a control procedure based on the approximate $A_x^{(j)}$'s in conjunction with the least-squares determination described above [11].

The approximation to the exact control law of Eq. (5) is constructed by transforming the measured $A_x^{(j)}$'s, $j<k_0$, to the quantities appearing on the left-hand side of the system of equations in Eq. (6), within the approximation outlined above. A time series is then generated at the nominal parameter value p_0 until it falls near $x=0$. At that time, the control is turned on and kept on as long as the required parameter perturbations are less than some prescribed value $\delta p_{\max}$, otherwise, the parameter is set back to p_0 until the trajectory falls near $x=0$ again. Even though x_i may be near $x\equiv \mathbf{P}_x(\boldsymbol{\xi}-\boldsymbol{\xi}_f)=0$, the observed x_{i+1} may be large, in the case where the full vector $\boldsymbol{\xi}_i$ is far from $\boldsymbol{\xi}_f$; this situation may occur since we assume only knowledge of the coordinate in the x direction. When the trajectory falls near $\boldsymbol{\xi}_f$, the x and p values will converge exponentially in time provided the k_0th approximation to Eq. (5) is sufficient. In cases where the number k_0 of coefficients $A_x^{(j)}$ that can be extracted from the data is insufficient for control, approximation techniques can be applied to extend the sequence of coefficients to $k>k_0$. Possible estimation methods will be discussed in a future publication [11]. It is important to note that the overall dimension of the phase space has no direct bearing either on the actual control procedure or on the difficulty of calculating the $A_x^{(j)}$'s.

We now examine how experimental errors are propagated in time once the feedback control is turned on. Assume that at time i, one observes $x_i=x_i^0+\eta_i$, where x_i^0 is the x value of the true system state and η_i is an uncorrelated noise term with $\langle\eta_i\rangle=0$. From the dynamics along the unstable coordinate u defined in Eq. (4) and the feedback rule in Eq. (5), one finds $|x_{i+1}|\le|x_{i+1}^0|+|\lambda_1 A_x^{(1)}/\mathbf{P}_x\mathbf{A}^{(u)}|\eta_{\max}$, where the noise distribution is bounded by $\eta_{\max}$. A large noise term may on occasion drive the dynamics out of the neighborhood of the fixed point where the dynamics is linear, and one must wait for the trajectory to return to the vicinity of ξ_f in order to reapply the control. In the presence of smaller errors, controlling the dynamics to always remain close to the fixed point is still possible. A similar result follows in the presence of noise originating in the time evolution of the system, as well as observational noise [11].

We illustrate our control procedure by applying it to the time evolution of a mechanical system composed of two connected rods known as the kicked double rotor [12]. By relating the state of the system after consecutive kicks of the system, the time development can be reduced to a four-dimensional dissipative mapping of the form [13] $\mathbf{X}_{i+1}=\mathbf{L}_1\mathbf{Y}_i+\mathbf{X}_i$ and $\mathbf{Y}_{i+1}=\mathbf{L}_2\mathbf{Y}_i+p\mathbf{G}(\mathbf{X}_{i+1})$, where $\mathbf{X}=(\theta_1,\theta_2)^T$ are the two angular position coordinates $(0\le\theta_i\le 2\pi)$, $\mathbf{Y}=(y_1,y_2)^T$ are the corresponding angular velocities, and $\mathbf{G}(\mathbf{X})$ is a nonlinear function. $\mathbf{L}_1$ and $\mathbf{L}_2$ are both constant matrices which involve the friction coefficients and moments of inertia of the rotor. The magnitude of the kick is p, which we shall utilize as the control parameter. At the constant nominal parameter value of $p_0=6.85$, and the other parameters as in Ref. [13], the system possesses a chaotic attractor. It is a two-piece attractor whose underlying periodic orbits consist only of even periods.

We focus on stabilizing a particular period-two cycle. Using a scalar time series of θ_1, the period-two orbit and its expanding eigenvalue are extracted. By embedding a measured scalar time series of length 10^4 in three dimensions, it was found that all the period-two orbits possess a single unstable direction. We choose to control the orbit with $\lambda_1=2.761$ and which alternatively visits the points

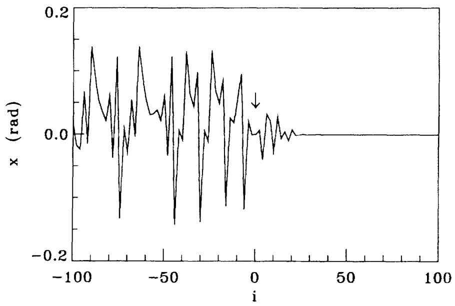

FIG. 1. The time variation of the measured signal $x \equiv \theta_1 - 2.113$ (every second iterate is shown) during the feedback control of a period-two orbit of the rotor ($p_0 = 6.85$, $\delta p_{max} = 0.05$, and $k_0 = 5$). The arrow indicates the time step at which perturbations were initiated.

with $\theta_1 = 2.113$ and $\theta_1 = 4.170$. For the former of these points, we found $A_x^{(i)} = -0.506$, 1.437, -0.554, 0.117, 0.017 for $i = 1, \ldots, 5$, respectively. These values were evaluated directly from a time series of θ_1, by measuring only every second time step, thus viewing the period-two orbit as a fixed point of the mapping composed twice. Utilizing the explicit form of the dynamical equations for the rotor, the $A_x^{(i)}$'s calculated according to the prescription following Eq. (3) agree closely with those extracted from the data. In Fig. 1, a typical time evolution of the measured value of $x \equiv \theta_1 - 2.113$ is shown for a feedback control based on Eq. (5) using $k_0 = 5$ (only even iterates are plotted). A portion of the uncontrolled chaotic trajectory at $p_0 = 6.85$ is also shown, corresponding to $i < 0$. The time at which the trajectory first enters a neighborhood of radius 10^{-2} of the periodic point ($\theta_1 = 2.113$) is denoted as $i = 0$. The parameter perturbations ($\delta p_{max} = 0.05$) were initiated at that time and the measured deviations x are seen to quickly converge to zero in Fig. 1. All the other phase-space coordinates also converged rapidly to the periodic orbit. In addition, the associated parameter perturbations decay to zero on the same time scale. Typically, a feedback which takes into account less of the $A_x^{(i)}$'s, i.e., $k_0 \leq 4$ in Eq. (5), was inadequate in controlling the dynamics around the period-two orbit from the measured values of θ_1, but may be sufficient for a different choice of measurement direction [11].

Cycles of order q may be controlled either by controlling the fixed points of the qth iterate of the dynamics (as was carried out above for the kicked double rotor with $q = 2$) or by applying a control of the form of Eq. (5) at each cycle point. In the latter alternative, it is necessary to determine a set of $A_x^{(i)}$'s for each of the q cycle points. For large q, this iterative fit procedure leads to improved control compared with carrying out a single fit for the qth iterate of the cycle, especially in the presence of noise [13]. The advantage of iterative fits for chaotic dynamics has been previously noted [7].

The control procedure can also be generalized to stabilize a periodic orbit with more than one local expanding direction. For such cases, the model of Eq. (3) would have coefficients $A_x^{(j)}$ which grow exponentially with j (the additional expanding directions contribute to these coefficients). In order to produce a convergent expansion which can be truncated, one must resort to embedding the scalar time series in a space whose dimension is equal to that of the unstable manifold. The corresponding form of Eq. (3) in the presence of more than one unstable eigenvalue will be given elsewhere [11]. In practice, there may be cases in which it is advantageous to carry out the feedback control by embedding a time series in a space whose dimension is somewhat higher than that of the dimension of the unstable manifold. In this way, the minimal k_0 at which the multidimensional generalization of Eq. (3) can be truncated may be reduced, without destroying control [11]. In general, there is an interplay between the dimension of the embedding space that is employed and the extent of the history that need be considered in order to achieve control.

This work was supported by the U.S. Department of Energy (Scientific Computing Staff Office of Energy Research). D.A. wishes to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada.

[1] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990).

[2] J. Singer, Y.-Z. Wang, and H. H. Bau, Phys. Rev. Lett. **66**, 1123 (1991); A. Azevedo and S. M. Rezende, Phys. Rev. Lett. **66**, 1342 (1991); E. R. Hunt, Phys. Rev. Lett. **67**, 1953 (1991); R. Roy, T. W. Murphy, Jr., T. D. Maier, Z. Gillis, and E. R. Hunt, Phys. Rev. Lett. **68**, 1259 (1992).

[3] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[4] G. Nitsche and U. Dressler, Phys. Rev. Lett. **68**, 1 (1992).

[5] T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990).

[6] See, for instance, R. Bowen, Trans. Am. Math. Soc. **154**, 377 (1991); P. Cvitanović, Phys. Rev. Lett. **61**, 2729 (1988); D. Auerbach, Ph.D. thesis, Weizmann Institute of Science, 1991 (unpublished).

[7] D. Auerbach, P. Cvitanović, J.-P. Eckmann, G. Gunaratne, and I. Procaccia, Phys. Rev. Lett. **58**, 2387 (1987).

[8] F. Takens, in *Dynamical Systems and Turbulence*, edited by D. A. Rand and L.-S. Young, Lecture Notes in Mathematics Vol. 898 (Springer, Berlin, 1981), p. 366.

[9] T. Sauer, J. A. Yorke, and M. Casdagli, J. Stat. Phys. **65**, 579 (1991).

[10] E. Ott, C. Grebogi, and J. A. Yorke, in *CHAOS, A Soviet-American Perspective in Nonlinear Science*, edited by D. Campbell (American Institute of Physics, New York, 1990).

[11] D. Auerbach, C. Grebogi, E. Ott, and J. A. Yorke (to be published).

[12] C. Grebogi, E. Kostelich, E. Ott, and J. A. Yorke, Physica (Amsterdam) **25D**, 347 (1987); Phys. Lett. A **118**, 448 (1986); **120**, 497(E) (1987).

[13] F. J. Romeiras, C. Grebogi, E. Ott, and W. P. Dayawansa, Physica D (to be published).