

### Mass Ratios of the Light Quarks

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We derive a consistent set of light quark mass ratios which are compatible with the available constraints of chiral symmetry treated to next-to-leading order. Also required as input is resonance saturation of a set of low-energy amplitudes, in particular for the pion and kaon electromagnetic mass difference. The results  $(m_d + m_u)/m_s = 1/15.5$ ,  $(m_d - m_u)/m_s = 1/29$  are modestly different from previous values, and resolve the long-standing puzzle of the  $\eta \rightarrow 3\pi$  decay rate.

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The masses of the  $u, d, s$  quarks are parameters in the Lagrangian of QCD. These small QCD (or current quark) masses are not to be identified with inertial (or constituent quark) masses and in fact generate only a small contribution to most physical quantities. Indeed we expect that the world would be largely unchanged if these QCD masses were to vanish. This feature poses special obstacles which stand in the way of any attempt to measure the masses. In practice, one generally uses symmetry relations between observables as a measure of mass effects—if a symmetry requires that two amplitudes are equal when the quarks are either massless (e.g., chiral symmetry) or identical in mass [e.g., isospin or flavor SU(3) symmetry], then the effect of nonzero mass or of mass splittings can be treated as a perturbation and the size of deviations from the symmetry limit can be used as a measure of the mass. By taking ratios of such deviations, any reduced matrix element often drops out, and one is left with an experimental measurement of the ratio of the quark masses themselves.

To first order in the quark masses, the results are well known [1,2]. On account of Goldstone's theorem applied to the chiral symmetry of QCD, pion and kaon masses must themselves vanish if the quark masses were set to zero [i.e., if chiral SU(3)<sub>L</sub> ⊗ SU(3)<sub>R</sub> symmetry were exact]. A linear expansion in quark mass then yields

$$\frac{m_u + m_d}{m_s + \hat{m}} = \frac{m_\pi^2}{m_K^2} = 0.075,$$

$$\frac{m_d - m_u}{m_s - \hat{m}} = \frac{m_{K^0}^2 - m_{K^+}^2 - m_{\pi^0}^2 + m_{\pi^+}^2}{m_K^2 - m_\pi^2} = 0.023;$$

i.e.,

$$\frac{\hat{m}}{m_s} = \frac{1}{26}, \quad \frac{m_d - m_u}{m_s} = \frac{1}{44}, \quad \frac{m_d - m_u}{m_d + m_u} = 0.27, \quad (1)$$

where  $\hat{m} = (m_u + m_d)/2$ . Since these results are derived using SU(3) symmetry, one should anticipate that they are valid to about 30%. Various authors have investigated second-order effects (see below). In this paper we dis-

cuss further the implications due to meson masses and propose a set of quark mass ratios valid at that order. Because the low-energy meson sector is the most tightly constrained and best understood, we restrict our treatment to this arena.

At second order in the quark masses, the analysis involves new reduced matrix elements, which have been completely classified within the context of the QCD effective chiral Lagrangian by Gasser and Leutwyler [3] and others [4,5], and which are described by the chiral coefficients  $L_i$  (we will need only  $L_7$  and  $L_{14}$ ). However, there emerges a fascinating subtlety [6]—via SU(3) symmetry one cannot distinguish the masses  $(m_u, m_d, m_s)$  from the set of masses  $m_u^{(\lambda)} = m_u + \lambda m_d m_s$ , plus cyclic permutations, for arbitrary  $\lambda$ . This reparametrization invariance [5–7] arises because both  $m_i$  and  $m_i^{(\lambda)}$  have the same SU(3) properties, so that either representation can be used in an SU(3) analysis to obtain the same physics, provided one allows the reduced matrix elements to change correspondingly. (For example,  $L_7 \rightarrow L_7^{(\lambda)} = L_7 + \lambda F_\pi^2/32 B_0$  with  $B_0 = -\langle 0 | \bar{\psi} \psi | 0 \rangle$ .) This  $\lambda$  invariance, of course, hinders attempts to measure quark mass ratios via symmetry relations.

There exist at least four separate constraints on quark mass ratios which have been analyzed to second order in the quark masses. Such calculations involve one loop chiral corrections, plus tree level contributions from the  $O(E^4)$  chiral Lagrangian. There are two sets of observables which are independent of the chiral coefficients. One utilizes the meson masses with electromagnetic self-energy effects removed [3]:

$$\left( \frac{m_d - m_u}{m_s - \hat{m}} \right) \frac{2\hat{m}}{m_s + \hat{m}} = \frac{m_\pi^2}{m_K^2} \frac{(m_{K^0}^2 - m_{K^+}^2)_{\text{QM}}}{m_K^2 - m_\pi^2},$$

$$(m_{K^0}^2 - m_{K^+}^2)_{\text{QM}} \equiv (m_{K^0}^2 - m_{K^+}^2)_{\text{EXPT}} - (m_{K^0}^2 - m_{K^+}^2)_{\text{EM}}. \quad (2)$$

The same combination of quark masses is derived from the  $\eta \rightarrow 3\pi$  decay amplitude [8],

$$\left( \frac{m_d - m_u}{m_s - \hat{m}} \right) \frac{2\hat{m}}{m_s + \hat{m}} = \frac{3\sqrt{3}F_\pi^2 \text{Re}A_{\eta \rightarrow \pi^+\pi^-\pi^0}(s_0)}{(m_K^2 - m_\pi^2)[1 + \Delta_{\eta 3\pi}]} \frac{m_\pi^2}{m_K^2}, \quad (3)$$

where  $A(s_0)$  indicates the amplitude at the center of the  $3\pi$  Dalitz plot and  $\Delta_{\eta 3\pi}=0.5$  is the chiral correction at  $O(E^4)$  which includes the effects of loop diagrams. [This result properly includes the effect of  $\eta$ - $\eta'$  mixing (we disagree with Ref. [9] which claims otherwise). All estimates of electromagnetic contributions put them at a level which is negligible.] Of course, consistency of the chiral expansion demands the numerical equality of both expressions. Use of the experimental rate  $\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0)=0.28 \pm 0.03$  keV implies that the right-hand side

$$\left( \frac{m_d - m_u}{m_d + m_u} \right) \frac{m_s + \hat{m}}{m_s - \hat{m}} = \frac{4}{3\sqrt{3}} \frac{\langle 0|F\bar{F}|\pi^0\rangle}{\langle 0|F\bar{F}|\eta\rangle} \frac{F_\pi}{F_\eta} \frac{F_K^2 m_K^2 - F_\pi^2 m_\pi^2}{F_\pi^2 m_\pi^2} (1 - \delta_{\text{GMO}}) \left[ 1 + \frac{4L_{14}}{F_\pi^2} (m_\eta^2 - m_\pi^2) \right] = 0.51 \left[ 1 + \frac{4L_{14}}{F_\pi^2} (m_\eta^2 - m_\pi^2) \right], \quad (5)$$

where  $L_{14}$  is a chiral coefficient related to the energy variation of the matrix elements of  $F_{\mu\nu}^A \tilde{F}^{A\mu\nu}$  and  $\delta_{\text{GMO}} = -0.06$ . While there may be some question about the use of the heavy quark methodology at the charm quark mass and/or the presence of possible higher-order operators [5,12,13], we will include this result in our analysis, as it is quite compatible with the other constraints. [It is also special because it goes beyond SU(3) symmetry and hence is not subject to the reparametrization transformation.] Finally, one has also the mass ratio [3]

$$\frac{2\hat{m}}{m_s + \hat{m}} = \frac{m_\pi^2}{m_K^2} [1 + \Delta_m], \quad (6)$$

$$\Delta_m = -0.43 - \frac{32L_7}{F_\pi^2} (m_K^2 - m_\pi^2),$$

where  $L_7$  is one of the chiral coefficients and  $\Delta_m$  represents the  $O(E^4)$  chiral correction. This relation changes with the reparametrization transformation.

The above relations are derived to be valid to second order. The lowest-order relation quoted in the introduction can be obtained from Eqs. (2) and (6) with  $\Delta_m=0$  and by use of Dashen's theorem [14]. This latter refers to a relation between pion and kaon electromagnetic mass splitting,  $(m_K^2 - m_\pi^2)_{\text{EM}} = m_{\pi^0}^2 - m_{\pi^+}^2$ , which is valid to zeroth order in the quark masses. When working beyond leading order, only the  $\eta \rightarrow 3\pi$  constraint, Eq. (3), can be applied without further work. However, it has not been common to use this relationship, but instead to focus on the pseudoscalar mass relations, Eqs. (2) and (6). In this regard, for example, one determination [3,7] continues to use Dashen's theorem at the next-to-leading order, and to employ the saturation of a sum rule for  $L_7$  by the  $\eta'$  resonance. This combination, when used in Eqs. (2) and (6),

of Eq. (3) is equal to  $2.35 \times 10^{-3}$ , and consequently that the kaon mass difference must be given by

$$(m_K^2 - m_{K^+}^2)_{\text{QM}} = 2m_K(7.0 \text{ MeV}), \quad (4)$$

to which we will return below. (There is also a related constraint from isospin breaking in  $K_{l3}$  decays [10]. While it is consistent with our analysis, we do not include it because the error bars are currently too large.) Note that the combination of masses in Eqs. (2) and (3) is unchanged under the reparametrization transformation.

A third observable can be obtained [5] via analysis of anomalous  $U(1)_A$  Ward identities and the heavy quark multipole expansion [11,12] to relate the transitions  $\psi' \rightarrow J/\psi \pi^0$  and  $\psi' \rightarrow J/\psi \eta$  to  $\pi$  and  $\eta$  matrix elements of  $F_{\mu\nu}^A \tilde{F}^{A\mu\nu}$ . Working to second order in the chiral expansion one finds

gives mass ratios essentially the same as the lowest-order results, Eq. (1). However, since Dashen's theorem yields  $(m_K^2 - m_{K^+}^2)_{\text{QM}} = 2m_K(5.3 \text{ MeV})$  instead of the value quoted in Eq. (4), there has persisted an inconsistency with the value obtained from  $\eta$  decay, and the  $\eta \rightarrow 3\pi$  rate has been considered to be a puzzling failure of chiral perturbation theory. Below we explore the physics behind the chiral mass relations and show that when one employs resonance saturation for all ingredients [ $L_7$ ,  $L_{14}$ , and  $(m_K^2 - m_{K^+}^2)_{\text{EM}}$ ] all constraints are compatible. A consistent set of mass ratios emerges, and the  $\eta \rightarrow 3\pi$  rate is no longer a problem.

In the past five years it has become clear that resonance saturation yields a remarkably accurate representation of the coefficient in chiral Lagrangians [3,15]. The physics behind this result is that the coefficients  $L_i$  parametrize the energy variation of low-energy matrix elements, which are known to be dominated by nearby resonance poles. In this way, the chiral effective Lagrangian methodology represents a '90s framework for the application of '60s insights on low-energy hadron physics, and this success motivates us to apply this technique to the analysis of pseudoscalar meson masses.

Both  $L_7$  and  $L_{14}$  are generated by the  $\eta'$ , as indicated by Leutwyler's analysis [7] and that of Ref. [5], producing  $L_7 = (-0.4 \pm 0.2) \times 10^{-3}$ ,  $L_{14} = (2.3 \pm 1.1) \times 10^{-3}$ . The case of  $L_7$  has some potential contamination by a  $\pi'$  resonance [13], and is exceptional in that it is the first time resonance saturation that has been applied to a parameter which changes under the SU(3) reparametrization transformation. However, if we allow some modest error bars on the quoted values, the inclusion of  $\eta'$  will suffice for our purposes.

It has long been known that use of vector meson dominated form factors in the Born diagrams yields an accurate estimation of the electromagnetic pion mass splitting [14–16], and below we extend this result in order to calculate the kaon electromagnetic mass difference. In the process of doing so, we find that the formalism of Socolow [16] is inconsistent with the low-energy constraints of chiral symmetry and does *not* reduce to that of Das *et al.* [17] in the soft pion limit. We have reanalyzed this problem and will present this now consistent formalism in a separate publication [18]. The resulting electromagnetic mass shifts are given by

$$\begin{aligned}
m_{\pi^0}^2 - m_{\pi^+}^2 &= \frac{2e^2}{F_\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2[(q+p)^2 - m_\pi^2]} \left\{ F_\pi^2(3q^2 + 4p \cdot q - 4p^2) - \frac{8F_\pi^2}{m_\rho^2 - q^2} \left[ 1 + \frac{1}{2} \frac{q^2}{m_\rho^2 - q^2} \right] \right. \\
&\quad \left. \times [p^2q^2 - (p \cdot q)^2] + 3q^2(q^2 + 2p \cdot q) \left[ \frac{F_\rho^2}{m_\rho^2 - q^2} - \frac{F_A^2}{m_A^2 - q^2} \right] \right\}, \quad (7) \\
m_{K^0}^2 - m_{K^+}^2 &= \frac{2e^2}{F_\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2[(q+p)^2 - m_K^2]} \\
&\quad \times \left\{ F_\pi^2(3q^2 + 4p \cdot q - 4p^2) - 8F_\pi^2[p^2q^2 - (p \cdot q)^2] \right. \\
&\quad \times \left[ \frac{2}{3} \frac{1}{m_\rho^2 - q^2} + \frac{1}{3} \frac{1}{m_\phi^2 - q^2} + \frac{1}{2} q^2 \left( \frac{1}{3} \frac{1}{(m_\rho^2 - q^2)^2} + \frac{2}{3} \frac{1}{m_\rho^2 - q^2} \frac{1}{m_\phi^2 - q^2} \right) \right] \\
&\quad \left. + 3q^2(q^2 + 2p \cdot q) \left[ \frac{2}{3} \frac{F_\rho^2}{m_\rho^2 - q^2} + \frac{1}{3} \frac{F_\phi^2}{m_\phi^2 - q^2} - \frac{F_{K_A}^2}{m_{K_A}^2 - q^2} \right] \right\},
\end{aligned}$$

where we have used the Weinberg sum rules [19]

$$\begin{aligned}
F_\pi^2 &= F_\rho^2 - F_A^2 = \frac{2}{3} F_\rho^2 + \frac{1}{3} F_\phi^2 - F_{K_A}^2, \\
F_\rho^2 m_\rho^2 &= F_A^2 m_A^2, \\
\frac{2}{3} F_\rho^2 m_\rho^2 + \frac{1}{3} F_\phi^2 m_\phi^2 &= F_{K_A}^2 m_{K_A}^2.
\end{aligned} \quad (8)$$

The mass shifts are finite when Eqs. (8) are obeyed. To lowest order in  $p^2$ , these results have also been obtained by Ecker *et al.* [15]. Evaluation of Eq. (7) yields  $\Delta m_\pi^2 = 2m_\pi(5.6 \text{ MeV})$  when the Kawarabayashi-Suzuki-Fayyazuddin-Riazuddin (KSFR) relation [20] ( $m_A^2 = 2m_\rho^2$ ) is used and  $\Delta m_\pi^2 = 2m_\pi(6.3 \text{ MeV})$  when the physical axial-vector meson mass is used [Expt =  $2m_\pi(4.6 \text{ MeV})$ ]. The formalism automatically respects Dashen's theorem at lowest order, but predicts a *significant violation* in the full calculation, with

$$\rho_{\text{EM}} \equiv \frac{(m_{K^0}^2 - m_{K^+}^2)_{\text{EM}}}{(m_{\pi^0}^2 - m_{\pi^+}^2)_{\text{EM}}} = 1.7 \quad (1.78), \quad (9)$$

where the first number uses KSFR masses and the second experimental axial-vector meson masses. (If we had blindly followed Socolow's approach [16] we would have obtained a very similar value,  $\rho_{\text{EM}} = 1.85$ .) The dominant SU(3) breaking in Eq. (9) arises from the kaon propagator in the Born diagrams. Since these should give the bulk of the electromagnetic mass difference [21], the violation of Dashen's theorem appears impossible to avoid [22]. The use of  $\rho_{\text{EM}} = 1.8 \pm 0.1$  together with the physical pion mass difference produces  $(m_{K^0}^2 - m_{K^+}^2)_{\text{QM}} = 2m_K(6.3 \pm 0.1 \text{ MeV})$ , which is within 10% of the value predicted from the  $\eta \rightarrow 3\pi$  relation, Eq. (4). From this

result we conclude that the  $\eta \rightarrow 3\pi$  constraint should *not* be discarded or regarded as a problem. Both  $\eta \rightarrow 3\pi$  and the kaon mass splitting yield essentially identical values for the  $m_d - m_u$  mass difference, once electromagnetic corrections are treated beyond leading order.

The chiral constraints, combined with the values of  $L_7$ ,  $L_{14}$ , and  $(\Delta m_K^2)_{\text{EM}}$  which were obtained through resonance saturation, are given in Table I. From these four inputs there exist two independent mass ratios, and we find that if one uses  $L_7 = -0.3 \times 10^{-3}$  and  $L_{14} = 1.2 \times 10^{-3}$ —within the ranges quoted above—then a consistent set of values is obtained:

$$\frac{m_d + m_u}{m_s + \hat{m}} = 0.061, \quad \frac{m_d - m_u}{m_s - \hat{m}} = 0.036;$$

i.e.,

$$\frac{\hat{m}}{m_s} = \frac{1}{31}, \quad \frac{m_d - m_u}{m_s} = \frac{1}{29}, \quad \frac{m_d - m_u}{m_d + m_u} = 0.59. \quad (10)$$

Compared to the lowest-order numbers [Eq. (1)] this amounts to a 20% decrease in  $\hat{m}/m_s$  and a 50% increase in  $(m_d - m_u)/m_s$ . The latter increase is to a large extent driven by Eq. (3) and should be a general feature of any set of mass ratios which is consistent with the  $\eta \rightarrow 3\pi$  constraint.

The chiral corrections to lowest-order relations involved in  $\eta \rightarrow 3\pi$  and in Dashen's theorem are rather large. However, this does not necessarily mean that the chiral expansion is breaking down. Similarly, large corrections have been found in  $I=0 \pi\pi$  scattering and in  $K \rightarrow \pi\pi/\nu$  [23] decays where, however, after accounting

TABLE I. Numerical values of quark mass ratios obtained from various sources. Note that only two of these are independent.

Mass ratio	Value	Source
$\left(\frac{m_d - m_u}{m_s - \hat{m}}\right) \frac{2\hat{m}}{m_s + \hat{m}}$	$2.11 \times 10^{-3}$	Meson masses and $(\Delta m_K^2)_{EM}$
$\left(\frac{m_d - m_u}{m_s - \hat{m}}\right) \frac{2\hat{m}}{m_s + \hat{m}}$	$2.35 \times 10^{-3}$	$\eta \rightarrow 3\pi$
$\left(\frac{m_d - m_u}{m_d + m_u}\right) \frac{m_s + \hat{m}}{m_s - \hat{m}}$	$0.67 \pm 0.16$	$\psi' \rightarrow J/\psi + \pi^0(\eta)$ and $L_7$
$\frac{2\hat{m}}{m_s + \hat{m}}$	$0.067 \pm 0.012$	Meson masses and $L_7$

for first-order corrections the results display a remarkable consistency. In the present problem, sizable corrections are required to bring the lowest-order results for  $\Delta m_K^2$  and  $\Gamma(\eta \rightarrow 3\pi)$  into agreement. It is encouraging that the physics behind these corrections is simple, and that various methods give a consistent extraction of the quark mass ratios.

In summary, aside from the ratio given from the  $\eta \rightarrow 3\pi$  amplitude, there is no analysis which uses symmetries alone, free of any additional assumptions, to extract quark masses at next-to-leading order. In previous treatments the constraints of Eqs. (3) and (5) were taken less seriously. Rather, Eqs. (2) and (6) were used plus the extra inputs of Dashen's theory beyond lowest order and an estimate of  $L_7$ . However, this leads to problems with  $\eta \rightarrow 3\pi$ . Here we have used the input of resonance saturation of various low-energy amplitudes in order to produce a consistent set of masses. A crucial requirement for obtaining this consistency is the violation of Dashen's theorem, i.e., a modification to the kaon electromagnetic mass shift beyond lowest order, which emerges from the consideration of the Born diagrams. The new mass values, Eq. (10), provide a plausible resolution to the long-standing problem of the  $\eta \rightarrow 3\pi$  decay rate within chiral perturbation theory.

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