Inverse Cascade and Wave Condensate in Mesoscale Atmospheric Turbulence

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It is shown that an inverse cascade of the turbulence of inertio-gravity waves produces a long-scale wave condensate. A new nonlinear equation is derived for long waves on rotating shallow water. It is proven that steady localized solutions are absent and that the condensate (a uniform inertial oscillation) is stable with respect to small perturbations. Wave self-interaction thus could not stop an inverse cascade of mesoscale geophysical turbulence. The implication of the existence of a condensate for the problem of tidal dissipation and retardation of the Earth's rotation is discussed.

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Studying long nonlinear waves on rotating shallow water seems to be of great importance for the physics of the atmosphere and the ocean. Mesoscale atmospheric and oceanic motions (with the scales larger than the medium depth H_0 and smaller than the planetary scale) could be described by the well-known shallow water equations in a rotating reference frame [1,2]. They are written in plane geometry for the horizontal velocity components $\vec{v}=(u, v)$ and for the depth of the medium h:

$$\frac{Du}{Dt}=-g\frac{\partial h}{\partial x}+fv,\quad \frac{Dh}{Dt}=-h(\vec{\nabla}\cdot\vec{v}),$$
$$\frac{Dv}{Dt}=-g\frac{\partial h}{\partial y}-fu,\quad \frac{D}{Dt}=\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla}). \tag{1}$$

The system (1) could also be thought of as describing a two-dimensional compressible gas, with h being the density.

The presence of the free boundary gives rise to the existence of waves (in addition to vortices). The system (1), after linearization, describes so-called inertio-gravity waves with the frequency spectrum

$$\omega_k^2=f^2+c^2k^2. \tag{2}$$

Short waves (with wavelengths much shorter than the Rossby radius $\rho=c/f$) are the usual gravity waves on shallow water with the speed $c=(gH_0)^{1/2}$. The Coriolis parameter f is the projection of the planetary vorticity 2Ω (with Ω being the rotation frequency) on the local vertical: $f=2\Omega\sin\phi$; ϕ is the local latitude.

These waves play an important role in a mesoscale circulation of the atmosphere [3] and of the ocean [2] (long tidal waves and storm surges). The atmospheric observational data (see [3,4] and references therein) for horizontal wind velocities show a remarkable universality of turbulence spectra at mesoscales (from tens to thousands of kilometers). A small-scale part (up to hundreds of kilometers) due to Kraichnan's inverse energy cascade [5] of two-dimensional vortex turbulence turns at larger scales into a steeper spectrum having a peak at $\omega\simeq f$, i.e., at the lowest frequency of the inertio-gravity waves. A sharp peak of the energy density at $\omega\simeq f$ has also been observed for tidal waves in the oceans [6]. This part of the energy spectrum in geophysical turbulent flows is evidently caused by waves, and its explanation (i.e., developing the consistent theory of the turbulence of inertio-gravity waves) is the subject of the present paper.

A natural way to study turbulence starts from finding conservation laws. Already the form of the dispersion relation (2) suggests that there should be a second motion integral besides the energy. The point is that three-wave interactions are forbidden: $\omega(k_1)+\omega(k_2)\neq\omega(\vec{k}_1+\vec{k}_2)$ for any $\vec{k}_1,\vec{k}_2$. It means that a weakly nonlinear wave dynamics is defined by four-wave scattering that does not change the total number of waves. To obtain this invariant, one should introduce canonical Hamiltonian variables that describe the waves. To be restricted by the dynamics of inertio-gravity waves, we take the potential vorticity $(\mathrm{curl}\,\vec{v}+f)/h$ to be a constant f/h_0 in space. This property is preserved with time since the potential vorticity is a Lagrangian invariant of the system (1). Under such a condition, one could describe the waves by introducing one pair of canonical variables (h,Φ) [7]. The velocity is expressed as follows:

$$u=\frac{\partial\Phi}{\partial x}-\frac{f}{h_0}\frac{\partial}{\partial y}\Delta^{-1}(h-h_0),$$
$$v=\frac{\partial\Phi}{\partial y}+\frac{f}{h_0}\frac{\partial}{\partial x}\Delta^{-1}(h-h_0).$$

Here Δ^{-1} is the inverse of the Laplacian operator. In these variables, Eqs. (1) turn into the simple Hamiltonian system $\partial h/\partial t=\delta H/\delta\Phi$, $\partial\Phi/\partial t=-\delta H/\delta h$, with the energy $H=\frac{1}{2}\int h(v^2+u^2+gh)d\vec{r}$ being a Hamiltonian. Making then the Fourier transform and introducing the normal wave amplitudes b_k by the formulas

$$h_k=(k^2H_0/2\omega_k)^{1/2}(b_k+b^*_{-k}),$$
$$\Phi_k=-i(\omega_k/2H_0k^2)^{1/2}(b_k-b^*_{-k}),$$

one can check that wave action $\int|b_k|^2d\vec{k}$ is an adiabatic invariant of the system (1). For a homogeneous turbulence, we introduce the pair correlation function $n_k\delta(\vec{k}-\vec{k}')=\langle b_kb^*_{k'}\rangle$. The total number of waves $\int n(\vec{k},t)d\vec{k}$ (the density of the wave action per unit volume) is a motion integral.

Having two motion integrals, the energy and the total number of waves, we could readily establish the directions of their fluxes by analogy with Fjortoft's theorem in 2D hydrodynamics (see, e.g., [1]). As it is usual under four-wave scattering, the energy goes downscale while the wave action cascades upscale by an inverse cascade. It has been shown [7] that a cascade energy transfer is impossible due to nonlocality so that the wave energy could reach small scales only due to a nonlocal transfer, which might correspond to the creation of fronts [8].

Here the inverse cascade is discussed. Steady turbulence spectra could be found analytically in the scale-invariant limits with either $k\rho \ll 1$ or $k\rho \gg 1$ when these spectra are power functions of the wave number. Usually to get spectra (up to a dimensionless factor) it is enough to use dimensional analysis of the different terms in the Hamiltonian [9]. However, some remarkable cancellations take place in the case in question so we calculate the Hamiltonian explicitly to get

$$H=\int \omega_k |b_k|^2 d\vec{k}+\tfrac{1}{2}\int V_{123}(b_1 b_2^* b_3^* +\text{c.c.})\delta(\vec{k}_1-\vec{k}_2-\vec{k}_3)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3$$
$$+\tfrac{1}{3}\int U_{123}(b_1 b_2 b_3+\text{c.c.})\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 .$$

Here ω_k is defined by (2) while the interaction coefficients are as follows:

$$U_{123}=U(\vec{k}_1,\vec{k}_2,\vec{k}_3)=\frac{k_1[\cos\theta_{23}(\omega_2\omega_3-f^2)+if\sin\theta_{23}(\omega_2-\omega_3)]+(1\leftrightarrow 2)+(1\leftrightarrow 3)}{(32H_0\omega_1\omega_2\omega_3)^{1/2}},$$

$$V_{123}=V(\vec{k}_1,\vec{k}_2,\vec{k}_3)=\{k_1[\cos\theta_{23}(\omega_2\omega_3-f^2)+if\sin\theta_{23}(\omega_2-\omega_3)]+k_2[\cos\theta_{13}(\omega_1\omega_3+f^2)+if\sin\theta_{13}(\omega_1+\omega_3)]$$
$$+k_3[\cos\theta_{21}(\omega_2\omega_1-f^2)+if\sin\theta_{21}(\omega_2+\omega_1)]\}/(8H_0\omega_1\omega_2\omega_3)^{1/2}.$$

Here θ_{ij} is the angle between the wave vectors $\vec{k}_i$ and $\vec{k}_j$.

Since three-wave processes are forbidden, one should make a quasilinear transformation $b_k \to a_k$ which eliminates nonresonant cubic terms from H and gives finally the interaction Hamiltonian $\int T_{1234}a_1 a_2 a_3^* a_4^* \delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4) \times d\vec{k}_1 \cdots d\vec{k}_4$ that describes four-wave scattering [9]. Here

$$T_{1234}=\frac{V_{1+212}^* V_{3+434}}{\omega_1+\omega_2-\omega_{1+2}}-\frac{U_{1+212}U_{3+434}}{\omega_{1+2}+\omega_1+\omega_2}-\left\{\frac{V_{131-3}^* V_{424-2}}{\omega_{4-2}+\omega_2-\omega_4}+(1,3\leftrightarrow 2,4)+(1,2\leftrightarrow 3,4)\right\},$$

with $V_{1+212}=V(\vec{k}_1+\vec{k}_2,\vec{k}_1,\vec{k}_2)$, etc. According to a general theory of wave turbulence [9], a steady spectrum carrying the action flux Q is as follows: $n_k \propto Q^{1/3} \times k^{-2m/3-2+\alpha/3}$. Here m and α are the scaling indices of T and ω, respectively.

At the small-scale limit, $\omega_k \to ck$ and the terms of order ck cancel each other in the denominators of T, so $m=4$, $\alpha=1$. The spectrum $n_k \propto Q^{1/3}k^{-13/3}$ thus found has been shown to be an exact solution for the respective kinetic equation, and it satisfactorily fits the data of atmospheric observations [7].

At the opposite limit ($k\rho \ll 1$), one has $m=2$, $\alpha=2$ and the spectrum is $n_k \propto Q^{1/3}k^{-8/3}$. Note that this type of spectrum can be observed in k space rather than in the frequency domain where it occupies a small interval $\omega - f \ll f$. The data accessible [2-4,6] are mostly from frequency spectra, and they do not allow a direct comparison with the spectrum obtained. Nevertheless, the data of [10] show the spectrum to become less steep as $\omega \to f$ in agreement with our result.

One could directly check that both spectra carry a constant flux of the waves towards large scales and naturally match each other at $k \simeq \rho^{-1}$. The inverse cascade of the wave turbulence thus proceeds until rather large scales, and this cascade gives an accumulation of the waves near the bottom of the frequency spectrum. To describe the destiny of the cascade, substantially nonlinear waves with the frequencies $\omega \approx f$ have to be considered (see below).

Note that the same problem of a large-scale sink for an inverse cascade led to the derivation of the Zakharov equation for plasma waves [11] as well as an application of the nonlinear Schrödinger equation (NSE) for optical turbulence [12]. In both cases, an extra motion integral (number of waves) exists causing an inverse cascade and the creation of a condensate that is a uniform wave field. Here the main question is the stability of the condensate. Wave systems are thus divided into two classes. The first one corresponds to an instability of the condensate and of the long waves. It happens if the sign of the nonlinear frequency shift is either negative (transverse instability) or opposite to that of the wave dispersion (longitudinal instability). Examples are given by Benjamin-Feir instability for water waves and modulational instability in plasma and optics. This takes place for the Zakharov equations and for the NSE with attraction between the waves. The instability destroys the condensate and provides the sink for an inverse cascade. For example, in plasma and in optics, a wave collapse (or self-focusing) is the result of that instability. Very-small-scale motions are created because of the collapse events and energy directly goes from large scales into a small-scale dissipative region [11,12]. The second class contains wave systems with a stable condensate as is the case for the NSE with repulsion. As a result of an inverse cascade, the condensate grows as well as the amplitudes of long waves. This causes the so-called Bogolyubov's renormalization of wave frequencies in the presence of a strong condensate (see [13]). The wave spectrum acquires a linear term ($\omega_k \propto k$ as $k \to 0$) which permits three-wave processes to be allowed and completely changes the picture of the large-scale turbulence [12].

Neither of those possibilities takes place in our case. To show that, let us obtain a nonlinear equation for the slow envelope of the wave with a carrier frequency f. First, we pass to the dimensionless variables $h/H_0, \vec{v}/c$ and $ft, r/\rho$. Then we introduce momentum $\vec{p}=h\vec{v}=(p,q)$ instead of velocity, so our system (1) is written as follows:

$$\frac{\partial p}{\partial t}-q+\frac{\partial}{\partial x}\left(\frac{p^2}{h}\right)+\frac{\partial}{\partial y}\left(\frac{pq}{h}\right)+h\frac{\partial h}{\partial x}=0,$$

$$\frac{\partial q}{\partial t}+p+\frac{\partial}{\partial y}\left(\frac{q^2}{h}\right)+\frac{\partial}{\partial x}\left(\frac{pq}{h}\right)+h\frac{\partial h}{\partial y}=0, \qquad (3)$$

$$\frac{\partial h}{\partial t}+\operatorname{div}\vec{p}=0.$$

Let us look for the solution of the system (3) in the form

$$h=1+\sum_{n=0}(h_n e^{-int}+\text{c.c.}),$$

$$p=\sum_{n=0}(p_n e^{-int}+\text{c.c.}),$$

$$q=\sum_{n=0}(q_n e^{-int}+\text{c.c.}).$$

We assume the amplitudes of the first harmonics h_1,p_1,q_1 to be much larger than h_n,p_n,q_n, respectively, for $n\neq 1$. We also assume all spatial scales to be much larger than unity. Our aim is to obtain a nonlinear equation for the amplitude of the first harmonic taking into account that nonzero amplitudes of other harmonics arise because of nonlinear interaction. Separating in (3) terms with different time exponents, we have in the main order

$$h_0=0,\quad p_0=-\frac{\partial|q_1|^2}{\partial y},\quad q_0=\frac{\partial|p_1|^2}{\partial x},$$

$$h_1=\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)q_1,\quad p_1=iq_1,$$

$$h_2=0,\quad p_2=iq_2,\quad q_2=\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)q_1^2.$$

Substituting that into the equations for $\partial p_1/\partial t$ and $i\partial q_1/\partial t$ and summing them up, after cumbersome calculations we obtain an elegant and compact equation for $\Psi=(q_1+ip_1)/2$:

$$2i\frac{\partial\Psi}{\partial t}+\Delta\Psi+2iJ(\Psi,|\Psi|^2)=0,$$

$$J(A,B)=\frac{\partial A}{\partial x}\frac{\partial B}{\partial y}-\frac{\partial A}{\partial y}\frac{\partial B}{\partial x}. \qquad (4)$$

Equation (4) is a kind of amplitude equation like the NSE and the Ginzburg-Landau equation. It is valid for scales much larger than the Rossby radius. For such scales, the dispersion of inertio-gravity waves is weak, of order $(k\rho)^2$. On the other hand, the scales are much smaller than the radius of the planet since we assume the Coriolis parameter f to be independent of coordinates. Nonlinearity was not assumed to be small while deriving (4) so that $|\Psi|$ can be larger than unity (which means that the Mach number is arbitrary). The smallness $|h-1|\ll 1$ has been assumed which is compatible with $\Psi\simeq 1$ due to a small dispersion. Equation (4) thus describes long nonlinear inertio-gravity waves in the atmosphere and tidal waves in the oceans (the same system could be also derived for cyclotron waves propagating perpendicularly to a strong magnetic field in a plasma).

As one can see, the Laplacian in (4) describes linear wave dispersion while the nonlinear term has the form of a Jacobian as is usual for two-dimensional incompressible flows. Indeed, for $k\to 0$ the height (or density) variation h is proportional to k, and it is small compared to the variations in the velocity field.

Let us stress that Eq. (4) seems to be quite nontrivial for nonlinear wave dynamics due to the form of the nonlinearity typical for the waveless situation. Such a form causes strong consequences which can be readily established. The condensate (that is, the solution $\Psi=$const, $v=\cos ft$, $u=\sin ft$, describing an inertial oscillation, like that of a Foucault pendulum, with a uniform fluid rotation opposite to that of the Earth) is linearly stable since it does not contribute to the nonlinear term; the perturbation spectrum is $\omega_k\propto k^2$. The same is true for plane and spherical waves [the latter is $\Psi(r,t)=e^{-i\Omega t}J_0(r\sqrt{2\Omega})$, with J_0 being the zeroth Bessel function]. The presence of the condensate does not change the frequencies of waves in the first order of perturbation theory. The point is that the nonlinearity could not change the frequency of uniform motion that is caused by an inertial Coriolis force acting on the moving particles. The presence of waves even of a large amplitude could hardly change the local rotation frequency (of the Earth, for instance). A nonlinear frequency shift could be thus in the dispersive part only. That shift is caused by the renormalization of the sound velocity $c=\sqrt{gh}$ due to the variation of h. The latter is small, so a nonlinear frequency shift could arise in the next order in small dispersion [terms $\propto(k\rho)^4$]. (It means, in particular, that even taking account of the next orders of the perturbation expansion does not change the spectrum of long perturbations $\omega_k\propto k^2$, unlike the case of Bogolyubov's renormalization).

Moreover, the following exact relation for the mean square radius of the distribution could be derived from (4):

$$\frac{\partial^2 R^2}{\partial t^2}=\frac{\partial^2}{\partial t^2}\int r^2|\Psi|^2dx\,dy=2\int|\nabla\Psi|^2dx\,dy>0. \qquad (5)$$

Therefore, steady solutions with finite R (regular and localized ones) are impossible in the framework of (4). Relation (5) demonstrates that any localized distribution spreads over the whole space so that the mean radius monotonically increases.

Steady vortex solutions of the Ginzburg-Pitaevsky type [12] with constant asymptotics at infinity (or something similar to dark solitons in the NSE with repulsion) are also absent in our case. Looking for a solution of the form $\Psi(r,\theta)=A(r)e^{im\theta+i\xi(r)}$, one could prove that $A \to \ln r$ as $r\to\infty$.

The stability of the condensate and the absence of a frequency renormalization and of localized solutions mean that the self-interaction of long waves could not provide a feedback for the inverse cascade. The cascade should go up until the interaction with planetary Rossby waves becomes essential. Rossby waves have frequencies that are much less than f so this interaction could be considered in adiabatic approximation. It means that the interaction with low-frequency waves could not violate the conservation of the number of high-frequency waves (which is an adiabatic invariant) and thus could not stop the inverse cascade. As a first approximation, we consider the influence of stationary geostrophic perturbations (of Rossby type) on the spreading of the packet of inertio-gravity waves.

Here we assume the presence of a small, slow perturbation of the height $h_0=1+\eta$, $\eta\ll 1$ which is in geostrophic balance [1–3] with the currents: $2p_0=-\partial h_0^2/\partial y$, $2q_0=\partial h_0^2/\partial x$. Substituting it into the equation for Ψ and neglecting other terms we get

$$2i\frac{\partial\Psi}{\partial t}+2iJ(1+\eta,\Psi)+(1+\eta)\Delta\Psi-\Psi\Delta\eta=0\,.$$

The absence of a bound state of $\Psi(x,t)$ in any one-dimensional well $\eta(x)$ could be established by direct analogy with perturbation theory in quantum mechanics [14]. Looking for the solution $\Psi(x,t)=e^{-iEt}\psi(x)$ and regarding energy E as a perturbation, one could reduce this problem to that of the linear Schrödinger equation with the potential $U(x)=\eta_{xx}/(1+\eta)$. Assuming a localized well ($\eta,\eta_x,\eta_{xx}\to 0$ as $x\to\pm\infty$) we see the potential to be positive

$$\int_{-\infty}^{\infty}U\,dx=\int_{-\infty}^{\infty}\frac{\eta_x^2}{(1+\eta)^2}dx>0\,.$$

It means the absence of bound states at least in one dimension. The presence of geostrophic modulations does not thus prevent the spreading of inertio-gravity wave packets.

Note that the fact obtained that the energy of mesoscale waves goes into the energy of stable almost uniform anticyclonic rotation may help to solve an old problem of the calculation of the observed retardation of the Earth's rotation. Tidal dissipation should be the main cause for this retardation [3]. The problem is that the paleontological data on the increase of the length of the day [15] as well as the study of satellite orbits [16] give a value of the dissipation rate about 5 times larger than the friction dissipation in shallow water. According to Broshe and Sundermann [17], the correct way is to calculate torques exerted by tidal currents rather than a scalar dissipation. The directions of the large-scale currents and the latitude at which they occur are thus relevant.

According to the present results, if the energy W is injected into the wave system at $\omega=2\Omega$ by tidal forces, then the part $Wf/2\Omega=W\sin\phi$ should be transferred into the energy of anticyclonic rotation. A corresponding part of the energy of storm surges should also go into the condensate. The higher the latitude the more energy comes to the condensate. On the other hand, the torque exerted by currents is proportional to $\cos\phi$. The main contribution stems thus from mid-latitudes. Account of real coastal geometry is needed for quantitative calculations. Though qualitative, the present concept of an inverse cascade and of condensate creation explains the existence of the large-scale anticyclonic currents that are necessary for the Broshe-Sundermann picture to be valid. Note that the presence of a large-scale anticyclonic flow in the equilibrium of a rotating fluid was already discussed (see [5,18] and references therein). Here we have shown how such a flow is produced by wave turbulence.

To conclude, we obtain the steady spectra of the turbulence of inertio-gravity waves and show that they cause the creation of the stable condensate. It explains the presence of a sharp peak at $\omega=f$ in the observational data [2–4,6].

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[1] M. Lesieur, *Turbulence in Fluids* (Kluwer, London, 1990).
[2] P. H. LeBlond and L. A. Mysak, *Waves in the Ocean* (Elsevier, Amsterdam, 1978), Vol. 20.
[3] A. Van Delden, Phys. Rep. **211**, 251–374 (1992).
[4] S. Badulin, Izv. AN SSSR, ser. Fiz. Atmos, i. Okeana **27**(6), 638 (1991) (English translation; Izv. Ac. Sci. USSR, Geophys. Ser.).
[5] R. H. Kraichnan and D. Montgomery, Rep. Progr. Phys. **43**, 548–619 (1980).
[6] B. Warren, in *Encyclopedia of Oceanography*, edited by R. Fairbridge (Reynhold, New York, 1966), pp. 590–596.
[7] G. Falkovich and S. Medvedev, Europhys. Lett. **19**, 279–284 (1992).
[8] D. Andrew and B. Hoskins, J. Atmos. Sci. **35**, 509 (1978).
[9] V. Zakharov, V. Lvov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Heidelberg, 1992), Vol. 1.
[10] G. Nastrom and K. Gage, Tellus Ser. A **35**, 383 (1983).
[11] V. Zakharov, Zh. Eksp. Teor. Fiz. **37**, 1417 (1972) [Sov. Phys. JETP **10**, 1005 (1972)].
[12] V. Dyachenko, A. Newell, A. Pushkarev, and V. Zakharov, Physica (Amsterdam) **57D**, 96–161 (1992).
[13] L. Landau and E. Lifshitz, *Statistical Physics II* (Pergamon, Oxford, 1982).
[14] L. Landau and E. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1982).
[15] W. H. Munk and G. MacDonald, *The Rotation of the Earth* (Cambridge Univ. Press, London, 1960).
[16] M. G. Rochester, EOS Trans. Am. Geophys. Union **54**, 769 (1973).
[17] P. Brosche and J. Sundermann, Pure Appl. Geophys. **86**, 95 (1971).
[18] J. Herring, J. Atmos. Sci. **34**, 1731-50 (1977).