Self-Similarity in Transient Stimulated Raman Scattering

C. R. Menyuk
Department of Electrical Engineering, University of Maryland, Baltimore, Maryland 21228-5398

D. Levi
Dipartimento di Fisica, Università di Roma, "La Sapienza," 00185 Roma, Italy

P. Winternitz
Centre de Recherches Mathématiques, Université de Montréal, Case Postale 6128-A, Montréal, Québec, Canada H3C 3J7
(Received 12 December 1991)

Stimulated Raman scattering in the transient limit is an integrable system. In contrast, however, to the usual behavior in integrable systems, solitons are transient and the behavior of the system at long distances is dominated by self-similar solutions which may be found by symmetry reduction. It is shown for fairly general initial conditions precisely which self-similar solution the system tends toward at long distances, and the system evolution is studied numerically. It is argued that this behavior in which self-similar solutions dominate the long-distance evolution should often appear in nonlinear optical systems with memory. A possible experiment is proposed.

PACS numbers: 42.65.Dr, 02.30.+g, 33.20.Fb

Stimulated Raman scattering in the transient limit in which pulse durations are short compared to the material deexcitation time T_2 (TSRS) has long been the focus of theoretical [1,2] and experimental [3,4] study. Recent experiments [4] have observed pump depletion and amplitude oscillation. Second Stokes generation imposed an upper limit on the length of these experiments, but it has since been found experimentally that it is possible to use a multiple pass geometry to suppress this generation [5], allowing one in principle to observe the TSRS system over very long lengths, as we shall later discuss. The equations which describe this system possess a Lax pair, and, hence, the system is integrable [6]. Yet, the qualitative behavior of this system is quite different from the "standard" behavior exhibited by integrable systems in one space and one time dimension, e.g., nonlinear light pulses in optical fibers, in which an initial pulse breaks up into some number of solitons and dispersive wave radiation so that solitons dominate the long-distance behavior. Solitons in stimulated Raman scattering are transient [7], and the long-distance behavior is dominated by self-similarity [2].

Both soliton solutions and self-similar solutions can be found by *symmetry reduction.* One looks for solutions of the original equations which, instead of depending on the space variable x and the time variable t separately, depend on a single variable ξ which combines x and t. For example, soliton solutions depend on the combination $\xi=x-Vt$, where V is the soliton velocity, while self-similar solutions depend on the combination xt. Powerful techniques, based on Lie algebra theory, allow one to determine the symmetry reductions for a given equation [8]. It has long been known that self-similar solutions, as well as soliton solutions, exist for integrable equations [9]. Why then have solitons received widespread attention, and been experimentally studied in a wide range of contexts, while the self-similar solutions are not well known, and have received little if any experimental attention? The reason is that in most contexts in which integrable equations appear, e.g., nonlinear light pulses in optical fibers, a pulse is introduced into a nonlinear medium, and the appropriate boundary conditions are that all fields tend toward zero at $t=\pm\infty$ if the propagation variable is x. Under these circumstances, solitons will dominate the behavior at large x.

Important exceptions exist in nonlinear optics, for example, three-wave interactions in which two of the waves are electromagnetic and one is a material excitation. This material excitation may be very long lived, in which case it is not appropriate to assume that it goes to zero as $t\rightarrow+\infty$. Physically, the key element appears to be *memory*—the ability of the medium to retain information long after the electromagnetic pulses have passed through. A number of examples already exist in the literature indicating that self-similar solutions will dominate the long-distance behavior when the system has memory. Manakov and co-workers [10] have studied light propagation in an inverted two-level medium and have found that a self-similar solution dominates the asymptotic behavior. An and Sipe [11] have found that Hill gratings evolve in a self-similar manner. Hilfer and Menyuk [2] showed numerically that self-similar behavior dominates the asymptotic evolution of stimulated Raman scattering in the transient regime. Recently, Levi, Menyuk, and Winternitz [12] showed that the self-similar solution can in general be expressed in terms of a Painlevé transcendent. If we demand that the solution be well behaved throughout the time domain, then the Painlevé transcendent is $P_{\rm III}$. This solution was first found by Elgin and O'Hare [13], and Tran and Haus [14] have recently used this solution to study the statistical properties of a system which starts from quantum noise. In all three cases—inverted two-level media, Hill gratings, and stimulated Raman scattering in gases—there is experi-

 © 1992 The American Physical Society

mental evidence cited in Refs. [2,3,10–12] that self-similarity dominates the long-term behavior. However, to our knowledge, self-similarity in optics has never been the subject of a careful experimental study.

In this Letter, we will focus on transient stimulated Raman scattering. We will give rules which determine the self-similar solution toward which a system will tend. These rules have been derived from inverse scattering theory, but we will present the details of the theory elsewhere since the details are complex and not physically illuminating [6]. Instead, we will present numerical results which indicate the correctness of the rules and, as importantly, give insight into how the self-similar solution is approached. At the end, we will describe an experiment which could verify the theoretical predictions. While the focus is on stimulated Raman scattering, we expect similar results to hold in other nonlinear optical systems in which the medium has memory. In addition to the examples previously given, we note that photorefractive materials in some cases obey equations identical to those of stimulated Raman scattering [15].

The equations which describe stimulated Raman scattering may be written after normalization [6,7] as

$$\partial A_1/\partial\chi = -A_2 X,\ \ \partial A_2 \partial\chi = A_1 X^*,\qquad \partial X/\partial\tau + \gamma X = A_1 A_2^*, \tag{1}$$

where χ and τ are normalized distance and time, A_1 and A_2 are normalized pump and Stokes amplitudes, X is proportional to the material excitation, and γ is inversely proportional to T_2, the material deexcitation time. From a physical standpoint, A_1 and A_2 are known as a function of time at the entry to the Raman cell and one then determines the subsequent evolution along the cell; mathematically, this corresponds to assuming that $A_1(\tau)$ and $A_2(\tau)$ are known at $\chi=0$ and determining $A_1(\chi,\tau)$ and $A_2(\chi,\tau)$ for $\chi>0$. When $\gamma=0$, a Lax pair exists and the system is integrable. It is this limit which we will be considering; it is relevant to experiments in the transient limit like those of Duncan *et al.* [4]. The quantity $K^2(\tau)=|A_1(\chi,\tau)|^2+|A_2(\chi,\tau)|^2$ does not change with χ. Assuming $\gamma=0$ and making the transformation

$$\tau' = \int_{-\infty}^{\tau} K^2(\tau'')d\tau''/T_\infty,\ \ \chi'=\chi T_\infty,\qquad A_1'=A_1/K(\tau),\ \ A_2'=A_2/K(\tau),\ \ X'=X/T_\infty, \tag{2}$$

where $T_\infty=\int_{-\infty}^{\infty}K^2(\tau'')d\tau''$, we find that the primed variables satisfy an equation identical to Eq. (1) but in which $|A_1'|^2+|A_2'|^2=1$ in the interval $0\le\tau\le 1$ and is zero elsewhere. We may thus assume with no loss of generality that our problem has this form. Physically, the new time variable corresponds to integrated intensity. We may similarly assume that A_1 is real for all τ at $\chi=0$, while A_2 is real at $\chi=0$, $\tau=0$, since transformations which rotate these complex angles do not change Eq. (1).

If we let $B_1(\xi)=A_1$, $B_2(\xi)=A_2$, and $Y(\xi)=\chi X$, where $\xi=\chi\tau$ is the similarity variable, then we obtain the ordinary differential equations [12,13]

$$dB_1/d\xi = -(1/\xi)B_2 Y,\ \ dB_2/d\xi=(1/\xi)B_1 Y^*,\qquad dY/d\xi = B_1 B_2^*. \tag{3}$$

These equations have nonsingular solutions which correspond to choosing the Stokes amplitude so that it is initially proportional to the pump amplitude. In these solutions, the phases of B_1 and B_2 are constant, corresponding to zero frequency mismatch between the pump and the Stokes beams. While Levi, Menyuk, and Winternitz [12] have found a more general set of self-similar solutions with nonzero frequency mismatches, one can show that if the initial pump and Stokes beams have no frequency mismatch, the most important case experimentally [4], then the system will always tend at long distances to one of the nonsingular solutions of Eq. (3). Thus, we will focus on these solutions. Their qualitative behavior is the same as has been observed numerically for a wide range of initial conditions in which the Stokes amplitude is initially small compared to the pump amplitude [2]. This evolution passes through three regimes: an I regime in which the Stokes amplitude grows exponentially while the pump remains nearly constant, a fully nonlinear transition regime, and a J regime in which the pump amplitude decays as $\xi^{-1/4}$ while undergoing a number of temporal oscillations proportional to $\xi^{1/2}$.

Writing $B_1=\cos[\beta_s(\xi)/2]$ and $B_2=\sin[\beta_s(\xi)/2]$, the self-similar solutions are parametrized by $\beta_{s0}\equiv\beta_s(\xi=0)$. They are also parametrized by an offset $\chi_{\rm off}$. Since Eq. (1) is not changed by replacing χ with $\chi+\chi_{\rm off}$, we could use as our similarity variable $\xi=(\chi+\chi_{\rm off})\tau$. We may write the pump and Stokes waves in the form

$$\{A_1(\chi,\tau)=\cos[\beta(\chi,\tau)/2]\exp[i\theta_1(\chi,\tau)],\qquad A_2(\chi,\tau)=\sin[\beta(\chi,\tau)/2]\exp[i\theta_2(\chi,\tau)]\},$$

where $\theta_1(\chi=0,\tau)=0$ and $\theta_2(\chi=0,\tau=0)=0$. Letting $\beta_0\equiv\beta(\chi=0,\tau=0)$ and $\beta_0'\equiv\partial\beta/\partial\tau$ evaluated at $\chi=0$, $\tau=0$, the rules which govern the self-similar solution toward which the system tends are as follows:

$$\beta_{s0}=\beta_0,\ \ \chi_{\rm off}=\beta_0'/(\sin\beta_0). \tag{4}$$

The behavior at large χ is completely dominated by the initial profile near $\tau=0$.

A justification of these rules based on inverse scattering theory will be given elsewhere. In this Letter, we confine ourselves to presenting numerical results which indicate the correctness of these rules and, just as importantly, yield insights into how the system approaches a self-similar solution. We recall that if $\beta(\chi=0,\tau)=\text{const}$, then the self-similar solution which is obtained by setting $\beta_{s0}=\beta_0$ and $\chi_{\rm off}=0$ exactly describes the evolution of Eq. (1) with $\gamma=0$. In Fig. 1, we compare the solution of Eq. (1) as a function of τ when $\chi=800$ to the solution of Eq. (3) up to $\xi=800$ when $\beta_0(\chi=0,\tau)=\beta_{s0}=0.2$. The solution to Eq. (1) is designated the TSRS (transient stimulated Raman scattering equation) solution, while the solution to Eq. (3) is designated the self-similar solution.

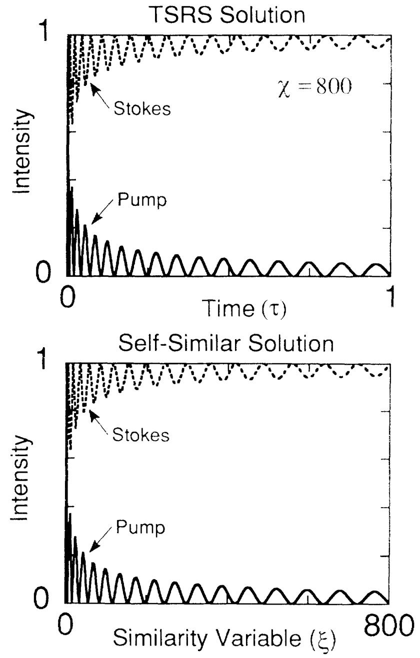

FIG. 1. The TSRS solution with $\beta(\chi=0,\tau)=\beta_0=0.2$ is compared to the self-similar solution with $\beta_{s0}=0.2$. They are identical.

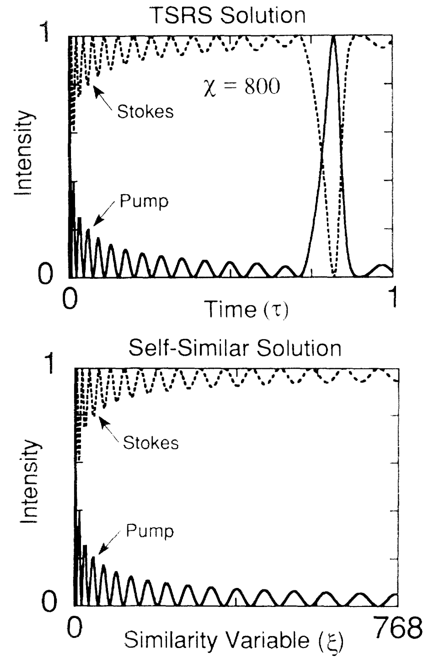

FIG. 3. The TSRS solution with $\beta(\chi=0,\tau)=0.2+2\pi\tau$ is compared to the self-similar solution with $\beta_{s0}=0.2$, $\chi_{\rm off}=-32$. A soliton is generated at the initial zero crossing. Agreement is good at times preceding the soliton.

As expected, the two solutions are identical. In Figs. 2 and 3, we show the evolution when $\beta(\chi=0,\tau)=0.2+2\pi\tau$ and $\beta(\chi=0,\tau)=0.2-2\pi\tau$, respectively, corresponding to $\beta_{s0}=0.2$ and $\chi_{\rm off}\simeq\pm 32$. In Fig. 2, we find that the TSRS solution compares quite well to the self-similar solution when the offset is taken into account and, indeed, detailed study of the evolution shows that the discrepancies are slight once $\chi \gtrsim 200$. By contrast, a large discrepancy can be seen in Fig. 3. A soliton was generated at the point $\tau_0=0.1/\pi$, where there was a zero crossing in the initial Stokes data, and, as χ increases, the soliton propagates to the back of the pulse and at $\chi \geq 1000$, it disappears. Nonetheless, when $\chi \gtrsim 200$, the TSRS solution resembles the self-similar solution at times which precede the soliton location, and, once the soliton has disappeared, agreement is excellent at all times. In Fig. 4, we show the solutions of Eq. (1), letting $\beta(\chi=0,\tau)=\beta_{s0}=0.2$ and $\theta_2(\chi=0,\tau)=10\pi\tau$. The initial phase wave rapidly disappears, and at the point $\chi=200$ solitons with phase shifts less than π have appeared. These propagate toward the back of the pulse at large τ and have for the most part disappeared at the point $\chi=800$. Excellent agreement was found between the TSRS solution shown in Fig. 4 at $\chi=800$ and the self-similar solution, although there are some discrepancies near $\tau=1$ due to a soliton. In summary, we found that after a period of transient behavior in which phase waves rapidly disappear and solitons eventually disappear, every TSRS solution which we examined tends toward a self-similar solution whose parameters are determined by the initial fields in the neighborhood of $\tau=0$ in accordance with Eq. (4).

To experimentally observe the predicted behavior in gases such as H_2, one must generate initial pump and Stokes profiles which are short compared to T_2. Duncan *et al.* [4] describe a method for obtaining 40-ps pump and Stokes pulses while T_2 in H_2 is typically ~ 1 ns. In this

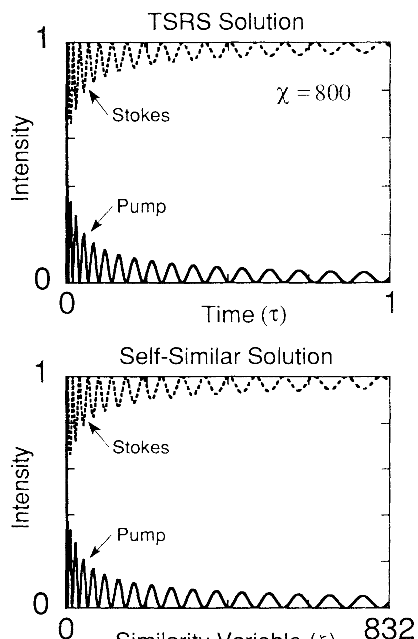

FIG. 2. The TSRS solution with $\beta(\chi=0,\tau)=0.2+2\pi\tau$ is compared to the self-similar solution with $\beta_{s0}=0.2$, $\chi_{\rm off}=32$. Agreement is good.

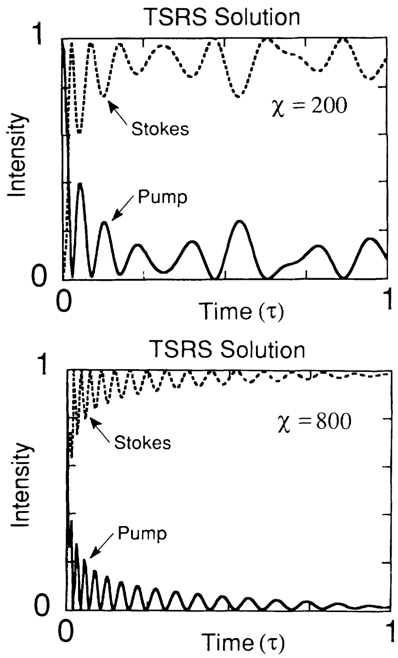

FIG. 4. The TSRS solution with $\beta(\chi=0,\tau)=\beta_0=0.2$, $\theta_2(\chi=0,\tau)=10\pi\tau$. Solitons with phase changes less than π are visible near $\tau=0.3$ and 0.7 at $\chi=200$, as verified by their motion toward larger times as χ increases. The soliton near $\tau=0.3$ propagates near to $\tau=0.9$ at $\chi=800$. Comparison to the self-similar solution in Fig. 1 shows that the agreement is good at times preceding the soliton.

experiment, self-similar oscillations were visible. To go the long length required to carry out a careful comparison of theory and experiment without obtaining second Stokes generation, one can use a multipass configuration like that described by MacPherson, Swanson, and Carlsten [5] which filters out the second Stokes radiation on each pass. A linear intensity gain of 30 is needed to obtain first Stokes radiation seeded from quantum noise when only the pump is present [4], and a similar intensity gain will lead to second Stokes radiation when sizable pump and first Stokes waves are present. This gain corresponds to $\chi\sim 50$. Since $\chi\sim 500$ is required to carry out careful measurements, 10 to 20 passes through the Raman amplifier are required, filtering out the second Stokes radiation on each pass. This choice will ensure that the growth of the second Stokes radiation is suppressed. Of course, it may be preferable to use a completely different medium, perhaps a photorefractive material, in which the second Stokes radiation is not phase matched.

In conclusion, we have shown that in the transient limit, under rather general conditions, the system of equations which describe stimulated Raman scattering tend toward a self-similar solution which is expressed in terms of $P_{\rm III}$, the third Painlevé transcendent. The self-similar solution toward which the system tends is entirely determined by the initial data at early times. We argue that this behavior will often appear in integrable, nonlinear optical systems with memory. We have numerically solved the original system of equations which describe stimulated Raman scattering and compared the results to the self-similar solutions, both to verify that the system always tends toward one of these solutions and to exhibit the transient soliton solutions which can appear in the early evolution. Finally, we have outlined an experiment which can be performed to study the phenomena which we have described.

C.R.M. gratefully acknowledges useful conversations with Dr. J. Carlsten and Dr. J. Reintjes. The work of C.R.M. was supported by the Naval Research Laboratory and by National Science Foundation Grant No. ECS-9114191. The work of D.L. and P.W. was supported by NSERC of Canada, FCAR of Quebec, INFN, and NATO. The numerical work was carried out at NERSC and SDSC.

[1] C. S. Wang, Phys. Rev. **182**, 482 (1969); R. L. Carman, F. Shimizu, C. S. Wang, and N. Bloembergen, Phys. Rev. A **2**, 60 (1970).

[2] C. R. Menyuk and G. Hilfer, Opt. Lett. **12**, 227 (1989); G. Hilfer and C. R. Menyuk, J. Opt. Soc. Am. B **7**, 739 (1990).

[3] M. E. Mack, R. L. Carman, J. Reintjes, and N. Bloembergen, Appl. Phys. Lett. **16**, 209 (1970).

[4] M. D. Duncan, R. Mahon, L. L. Tankersley, and J. F. Reintjes, J. Opt. Soc. Am. B **5**, 37 (1988).

[5] D. C. MacPherson, R. C. Swanson, and J. L. Carlsten, IEEE J. Quantum Electron. **25**, 1741 (1989).

[6] F. Y. F. Chu and A. C. Scott, Phys. Rev. A **12**, 2060 (1975); D. J. Kaup, Physica (Amsterdam) **6D**, 143 (1983); D. J. Kaup and C. R. Menyuk, Phys. Rev. A **42**, 1712 (1990).

[7] C. R. Menyuk, Phys. Rev. Lett. **62**, 2937 (1989).

[8] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986); P. Winternitz, in *Partially Integrable Evolution Equations in Physics*, edited by R. Conte and N. Boccara (Kluwer, Dordrecht, 1990), pp. 517–567.

[9] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981), pp. 67–83.

[10] S. V. Manakov, Zh. Eksp. Teor. Fiz. **83**, 68 (1982) [Sov. Phys. JETP **56**, 37 (1982)]; I. R. Gabitov and S. V. Manakov, Phys. Rev. Lett. **50**, 495 (1983); I. R. Gabitov, V. E. Zakharov, and A. V. Mikhailov, Zh. Eksp. Teor. Fiz. **86**, 1204 (1984) [Sov. Phys. JETP **59**, 703 (1984)].

[11] S. An and J. E. Sipe, Opt. Lett. **16**, 1478 (1991).

[12] D. Levi, C. R. Menyuk, and P. Winternitz, Phys. Rev. A **44**, 6057 (1991).

[13] J. N. Elgin and T. B. O'Hare, J. Phys. B **12**, 159 (1979).

[14] T. K. Tran and J. W. Haus, Phys. Rev. A **44**, 4608 (1991).

[15] N. Kukhtarev, V. Markov, and S. Odulov, Opt. Commun. **23**, 338 (1977); J. M. Heaton and L. Solymar, IEEE J. Quantum Electron. **24**, 558 (1988).