PHYSICAL REVIEW
LETTERS

VOLUME 69 | 6 JULY 1992 | NUMBER 1

Time Reversal in Stochastic Processes and the Dirac Equation

D. G. C. McKeon and G. N. Ord
Department of Applied Mathematics, University of Western Ontario, London, Canada N6A 5B7
(Received 6 January 1992)

We consider the motion of a classical particle in $(1+1)$-dimensional space-time. Four probability distributions govern the trajectory of the particle; these give the probability of moving to the left or right in space while moving backwards or forwards in time. If these probabilities are randomly distributed and if the probability of moving backwards in time is related to the probability of moving forwards in time in a prescribed manner, then the master equations for these probabilities give rise to the Dirac equation without recourse to direct analytic continuation. In contrast, when a particle always moves forward in time, an analytic continuation is required to recover the Dirac equation.

PACS numbers: 03.65.Ca

Gaveau *et al.* [1] have considered the random motion of a particle in $(1+1)$-dimensional space-time. The trajectory of the particle is always forward in time and consists of a sequence of reversals in spatial motion. The probability of such a reversal in time Δt is $a\Delta t$, so that the master equation governing the particle's motion is

$$P_\pm(x,t+\Delta t)=(1-a\Delta t)P_\pm(x\mp\Delta x,t)+a\Delta tP_\mp(x\pm\Delta x,t). \qquad (1)$$

Here $P_\pm(x,t)$ is the probability of a particle being at (x,t), moving either to the right $(+)$ or left $(-)$. It is shown in [1] how in the limit $\Delta x,\Delta t\to 0$ (with $\Delta x/\Delta t$ = finite) the Dirac equation can be recovered provided a is continued to an imaginary value [or, equivalently, $(t,\Delta t)$ is continued to $(it,i\Delta t)$].

We extend these considerations to the case where the classical particle can move both backwards and forwards in time. There are now four probability distributions: $F_\pm(x,t)$ and $B_\pm(x,t)$, where B and F refer to "backwards" and "forwards" in time. It is also possible to distinguish between the probability of turning "left" ($a_L\Delta t$) and "right" ($a_R\Delta t$) in space-time in time Δt. It is now evident that (1) generalizes to

$$\begin{aligned}F_\pm(x,t)=&(1-a_L\Delta t-a_R\Delta t)F_\pm(x\mp\Delta x,t-\Delta t)\\&+a_{L,R}\Delta tB_\pm(x\mp\Delta x,t+\Delta t)\\&+a_{R,L}\Delta tF_\mp(x\pm\Delta x,t-\Delta t).\end{aligned} \qquad (2)$$

We now impose the condition that

$$F_\pm(x,t)=B_\mp(x\pm\Delta x,t+\Delta t). \qquad (3)$$

By (3), (2) implies that

$$\begin{aligned}B_\pm(x\mp\Delta x,t+\Delta t)=&(1-a_L\Delta t-a_R\Delta t)B_\pm(x,t)\\&+a_{L,R}\Delta tB_\mp(x,t)\\&+a_{R,L}\Delta tF_\pm(x,t).\end{aligned} \qquad (4)$$

We note that (4) is not obtained from (2) by merely performing a "rotation" in the x-t plane; that would lead to

$$\begin{aligned}B_\pm(x,t)=&(1-a_L\Delta t-a_R\Delta t)B_\pm(x\mp\Delta x,t+\Delta t)\\&+a_{L,R}\Delta tB_\mp(x\pm\Delta x,t+\Delta t)\\&+a_{R,L}\Delta tF_\pm(x\mp\Delta x,t-\Delta t).\end{aligned} \qquad (5)$$

Iteration of (1) leads to an interpretation in terms of a "random walk"; we see that

$$\begin{aligned}P_+(x,t)&=(1-a\Delta t)P_+(x-\Delta x,t-\Delta t)+a\Delta tP_-(x+\Delta x,t-\Delta t)\\&=(1-a\Delta t)\{(1-a\Delta t)P_+(x-2\Delta x,t-2\Delta t)+a\Delta tP_-(x,t-2\Delta t)\}\\&\quad+a\Delta t\{(1-a\Delta t)P_-(x+2\Delta x,t-2\Delta t)+a\Delta tP_+(x,t-2\Delta t)\}\\&=\cdots.\end{aligned} \qquad (6)$$

This can be continued n times, until finally $P_+(x,t)$ can be expressed in terms of $P_\pm$ evaluated at $t-n\Delta t$. We assume that at that time the particle's state is known [e.g., $P_+(x+k\Delta x,t-n\Delta t)=1$ and $P_\pm=0$ at time $t-n\Delta t$ otherwise]. It is evident that (6) implies

$$P_+(x,t)=\sum_{\text{paths}}(1-a\Delta t)^{n-R}(a\Delta t)^R\,, \tag{7}$$

where we sum over all paths from $(x+k\Delta x,t-n\Delta t)$ to (x,t) beginning and ending by moving to the right, and always in the direction of increasing time. The number of turns in each path is R. If $a\Delta t \simeq i\varepsilon$ and $(1-a\Delta t)\simeq 1$, then (7) reduces to the random walk for the *amplitude* ϕ described in [2].

Iteration of (2) and (4) is not nearly so straightforward; however, it is easily seen that $F_\pm(x,t), B_\pm(x,t)$ depend only on probabilities that lie in the past. This consistency with causality does not follow from (2) and (5). In this sense the condition of (3) is "causal."

In the limit $\Delta x,\Delta t\to 0$ with $\Delta x=v\Delta t$, (2) and (4) yield

$$\pm v\frac{\partial F_\pm}{\partial x}+\frac{\partial F_\pm}{\partial t}=a_{L,R}(-F_\pm+B_\pm)+a_{R,L}(-F_\pm+F_\mp)\,, \tag{8a}$$

$$\pm v\frac{\partial B_\mp}{\partial x}+\frac{\partial B_\mp}{\partial t}=a_{L,R}(-B_\mp+F_\mp)+a_{R,L}(-B_\mp+B_\pm)\,. \tag{8b}$$

If $A_\pm=\exp[(a_L+a_R)t](F_\pm-B_\mp)$, then (8) becomes

$$v\frac{\partial A_\pm}{\partial x}\pm\frac{\partial A_\pm}{\partial t}=\lambda A_\mp \quad (\lambda\equiv -a_L+a_R)\,. \tag{9}$$

The equation

$$\pm v\frac{\partial z_\pm}{\partial x}+\frac{\partial z_\pm}{\partial t}=\rho z_\mp \quad (\rho\equiv a_L+a_R) \tag{10}$$

results if we were to consider $z_\pm=\exp[(a_L+a_R)t]\times(F_\pm+B_\mp)$.

We note that (10) is essentially the same as (9), except that the roles of x and t are reversed. Consequently, if we square (10), the Klein-Gordon equation results except that the mass term has the wrong sign, leading to exponentially growing or decaying solutions. We therefore exclude (10). Physically, this is reasonable since an observer moving forward in time would interpret the probabilities $B_\pm$ as being associated with the trajectories of antiparticles, and the difference $F_\pm-B_\mp$ is naturally interpreted with the net flux of particles moving forward in time. (This is a form of "charge conservation.")

Upon setting $v=c$, $\lambda=mc^2/\hbar$, and $\psi^T=(A_+,A_-)$, (9) can be written as

$$i\hbar\frac{\partial\psi}{\partial t}=mc^2\sigma_y\psi-ic\hbar\sigma_z\frac{\partial\psi}{\partial x}\,. \tag{11}$$

This is the Dirac equation in $(1+1)$-dimensional Minkowski space; the corresponding Weyl equation (i.e., the $m\to 0$ limit) arises if $\lambda=-a_L+a_R$ is zero. No direct analytic continuation of λ is required in this derivation. This is possible as there is in two dimensions a representation of the Dirac matrices in $1+1$ dimensions in which the free wave function is real; the wave function does not remain real in this representation if there is an external vector potential.

We thus see that the random motion of a particle in $(1+1)$-dimensional space directly leads to the Dirac equation provided motion backwards in time is incorporated using (3). It is essential to have both probabilities $F_\pm$ and $B_\pm$ to get to this result. A similar conclusion has been obtained using the transfer matrix approach [3].

We hope to extend this discussion beyond $1+1$ dimension using the techniques of [1].

NSERC provided financial support. R. and D. MacKenzie had a helpful suggestion.

[1] B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman, Phys. Rev. Lett. **53**, 419 (1984).

[2] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), p. 35.

[3] G. N. Ord, Int. J. Theor. Phys. (to be published).