Relaxation Towards a Statistical Equilibrium State in Two-Dimensional Perfect Fluid Dynamics

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In previous works we have defined statistical equilibrium states for two-dimensional incompressible Euler equations. We establish here evolution equations governing the relaxation of the system towards these equilibrium states.

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It is well known that 2D slightly viscous flows tend to form coherent structures. In previous work [1-6], we have developed an equilibrium statistical mechanics which aims to predict and describe these structures as steady states for 2D incompressible Euler equations. (An equivalent theory, using a less rigorous mean-field approach, has been derived independently by Miller [7]. A comparison with Miller's approach is made by Robert [3].) Actually, the resolution of the stationary equation which gives the equilibrium states is a complex program since a wide range of bifurcations may occur [5,6,8]. Furthermore, the Lagrange multipliers arising as unknown parameters in this equation are not easily expressed in terms of the constants of the motion of the initial datum.

We establish here a set of time-dependent equations which govern the relaxation of the system towards its equilibrium state. Our aim is twofold. First, it is to provide a convenient algorithm to reach the equilibrium state (or the equilibrium set when it does not reduce to a unique state) corresponding to a given initial datum. Second, it can be used as a sub-grid-scale modeling: It might describe the complex evolution of the flow without having to handle a detailed description of the small scales. Our relaxation equations are of a diffusion-convection type. The main difference from Navier-Stokes equations is that they are designed to conserve the energy and all the constants of the motion of Euler equations. To close the set of equations we propose a variational principle for the diffusion fluxes: For a given rate of entropy production the system tends to minimize the diffusion energy of the fluxes (or equivalently for a given diffusion energy it tends to maximize its entropy production). Although this approach yields an efficient way to reach the equilibrium state (we prove that if the solution converges towards some state, it is indeed a Gibbs state), we are not *a priori* guaranteed to mimic the actual dynamics of the Euler flow. Of course this issue has to be carefully tested.

We start here with Euler equations in an open, bounded, simply connected and regular domain Ω of the plane. Let $\mathbf{u}(t,\mathbf{x})$ be the velocity field; the incompressibility condition is solved by introducing the stream function $\psi(t,\mathbf{x})$. We consider the scalar vorticity $\omega(t,\mathbf{x})=(\nabla\times\mathbf{u})\cdot\mathbf{k}$, with $\mathbf{k}$ the unit vector normal to the plane, and write the Euler system

$$\omega_t+\nabla\cdot(\omega\mathbf{u})=0,\quad \omega(0,\mathbf{x})=\omega_0(\mathbf{x})\,, \tag{1}$$

$$\mathbf{u}=\nabla\times(\psi\mathbf{k}),\quad \omega=-\nabla^2\psi,\quad \psi=0 \text{ on } \partial\Omega\,. \tag{2}$$

For any bounded initial vorticity $\omega_0(\mathbf{x})$, the system (1),(2) has a unique bounded solution $\omega(t,\mathbf{x})$ [9]. We assume here that the initial condition is made of patches with n uniform vorticity levels a_i (but generalization to a continuous vorticity distribution is straightforward in the limit $n\to\infty$). All the known constants of the motion in our domain are the following functionals: the energy

$$E\{\omega\}=\frac{1}{2}\int_\Omega \mathbf{u}^2 d\mathbf{x}=\frac{1}{2}\int_\Omega \psi\omega\, d\mathbf{x}\,;$$

the area $|\Omega^i|$ of each vorticity patch Ω^i with uniform value a_i; if Ω is a disk $B(O,R)$, the angular momentum $\mathbf{M}\{\omega\}$ with respect to O, $\mathbf{M}\{\omega\}=\int_\Omega \mathbf{x}\times\mathbf{u}(\mathbf{x})d\mathbf{x}=[\frac{1}{2}\int_\Omega(R^2-\mathbf{x}^2)\omega(\mathbf{x})dx]\mathbf{k}$.

After some evolution, the solution of the Euler equations becomes in general extremely complicated. Instead of a detailed description of the vorticity field, we introduce the macroscopic variables $p_i(\mathbf{x})$, $i=1,\ldots,n$, which give, at each point $\mathbf{x}$, the probability of finding the value a_i. It has been proved [2,3] that an overwhelming majority of all the vorticity fields with given constants of the motion are close to a macroscopic state (the equilibrium state), or to a set of such states (the equilibrium set). These states are obtained by maximizing the mixing entropy

$$S\{\mathbf{p}\}=\int_\Omega s(\mathbf{p})d\mathbf{x},\quad \mathbf{p}=(p_1(\mathbf{x}),\ldots,p_n(\mathbf{x})),$$
$$s(\mathbf{p})=-\sum_i p_i\ln p_i\,, \tag{3}$$

under the constraints (i)

$$\sum_i p_i(\mathbf{x})=1,\ \text{for all } \mathbf{x}\,,$$

(ii)

$$F_i\{\mathbf{p}\}=\int_\Omega p_i(\mathbf{x})d\mathbf{x}=|\Omega^i|,\quad i=1,\ldots,n\,,$$

(iii)

$$E\left\{\sum_i a_ip_i(\mathbf{x})\right\}=E\{\omega_0\}\,.$$

It was shown that this problem, to which we will refer as (V.P.) in the following, always has a solution (possibly not unique). If $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$ is a solution of (V.P.) such that each function $p_i^*(\mathbf{x})$ is continuous and strictly positive on Ω, we can show that there exist Lagrange multipliers $\alpha_1, \ldots, \alpha_n, \beta$ such that

$$p_i^*(\mathbf{x}) = \frac{\exp[-\alpha_i - \beta a_i \psi^*(\mathbf{x})]}{Z(\psi^*(\mathbf{x}))}, \quad i=1,\ldots,n, \qquad (4)$$

where $Z(\psi) = \sum_i \exp(-\alpha_i - \beta a_i \psi)$, and ψ^* is the stream function associated with the locally averaged vorticity $\omega^* = \sum_i a_i p_i^*$. As a result of the relationship $\sum_i p_i(\mathbf{x}) = 1$, the functionals $F_1, \ldots, F_n$ give only $n-1$ independent constraints, and we can always take $\alpha_n = 0$. Thus to find the equilibrium states, we must solve the nonlinear elliptic equation [10]:

$$-\nabla^2 \psi = -\frac{1}{\beta}\frac{d}{d\psi}\ln Z, \quad \psi = 0 \text{ on } \partial\Omega. \qquad (5)$$

It always has a unique solution when β is greater than some negative value β_c, but when $-\beta$ is sufficiently large, bifurcations to multiple solutions generally occur [6,8].

We shall assume that during its evolution towards a final equilibrium state, the system can already be described in terms of a set of local probabilities $p_1(t,\mathbf{x}), \ldots, p_n(t,\mathbf{x})$. In other words, the system has already undergone fine-scale vorticity oscillations, and the velocity field $\mathbf{u}(t,\mathbf{x})$ is obtained by the integration of (2), where $\omega(t,\mathbf{x})$ is replaced by the locally averaged vorticity $\bar{\omega}(t,\mathbf{x}) = \sum_i a_i p_i(t,\mathbf{x})$. The vorticity patches are transported by this velocity field, and we suppose that in addition they undergo a diffusion process, so that the conservation equation for each vorticity probability can be written

$$(p_i)_t + \nabla\cdot(p_i \mathbf{u} + \mathbf{J}_i) = 0, \quad i=1,\ldots,n, \qquad (6)$$

$\mathbf{J}_i$ is the diffusion flux of the patch i. We impose the boundary condition $\mathbf{J}_i \cdot \mathbf{n} = 0$, so that the total area occupied by each patch is conserved. We can assume (without loss of generality) $\sum_i \mathbf{J}_i = 0$, i.e., the locally averaged velocity of the fluid is $\mathbf{u}(t,\mathbf{x})$. Denoting the vorticity flux $\mathbf{J}_\omega = \sum_i a_i \mathbf{J}_i$, we deduce from (6) an equation for the locally averaged vorticity

$$\bar{\omega}_t + \nabla\cdot(\bar{\omega}\mathbf{u} + \mathbf{J}_\omega) = 0, \qquad (7)$$

Since the velocity field is entirely determined by the field $\bar{\omega}$, the total energy is

$$E\{\bar{\omega}\} = \tfrac{1}{2}\int_\Omega \psi\bar{\omega}\, d\mathbf{x}. \qquad (8)$$

In other words we neglect the diffusion energy $E_d = \frac{1}{2}\int_\Omega \sum (\mathbf{J}_i^2/p_i) d\mathbf{x}$, associated with the diffusion transport, in front of $E\{\bar{\omega}\}$. Let us compute the rate of change of energy in the convection-diffusion process (6). An integration by part gives $\dot{E}\{\bar{\omega}\} = \int_\Omega \psi\bar{\omega}_t\, d\mathbf{x}$. Using (7), we find

$$\dot{E} = \int_\Omega \nabla\psi\cdot \mathbf{J}\omega\, d\mathbf{x}. \qquad (9)$$

We need also to compute the rate of entropy production

$$\dot{S} = \int_\Omega \sum_i s_i \times (p_i)_t\, d\mathbf{x} = -\int_\Omega \sum_i \nabla p_i \cdot (\mathbf{J}_i/p_i) d\mathbf{x}, \qquad (10)$$

where $s_i = \partial s/\partial p_i = -1 - \ln p_i$.

To get a closed set of equations, we need to relate the fluxes $\mathbf{J}_i$ to the probability fields p_i. The usual procedure is to assume that a local equilibrium is achieved at each time, and to impose a linear relationship between diffusion fluxes and gradients of the local thermodynamic intensive parameters (like temperature), such that the system is driven towards the final equilibrium state with monotonous increase of entropy. In our case it is difficult to define a local equilibrium. In short, this is due to the fact that the entropy density $s(\mathbf{p})$ is a function of the p_i's alone. There is no local internal energy, so that the parameter β is not locally defined. This is why we shall make the hypothesis that the system is near the *global* equilibrium. We present first the method in the case with only two initial vorticity levels a and 0. The state of the system is then completely determined by the probability $p_1(\mathbf{x})$ of finding the level a, or equivalently by the locally averaged vorticity $\bar{\omega}(\mathbf{x}) = a p_1(\mathbf{x})$. At equilibrium, $s' - \beta\psi = \alpha_1/a$ is a constant on Ω ($s' = ds/d\bar{\omega}$); this is just equivalent to (4). Thus, near equilibrium, this quantity will have only weak gradients. Since there is no internal energy, the relaxation towards equilibrium is only controlled by the diffusion of vorticity. Therefore we can write a linear relationship between $\mathbf{J}_\omega$ and $\nabla(s' - \beta\psi)$. As a simple approach, we make the strong assumption that this relationship is local and isotropic: $\mathbf{J}_\omega = \mathcal{A}(\mathbf{x})\nabla(s' - \beta\psi)$. But $\nabla s' = -\nabla\bar{\omega}/[\bar{\omega}(a-\bar{\omega})]$, and to avoid singularities $\mathcal{A}(\bar{\omega})$ must vanish for $\bar{\omega} = 0$ and $\bar{\omega} = a$. Then the simplest choice is $\mathcal{A}(\bar{\omega}) = \bar{\omega}(a-\bar{\omega})A$, where A is a constant. The parameter β is determined by the conservation of energy, using (9), which yields

$$\beta = -\int_\Omega \nabla\psi\cdot\nabla\bar{\omega}\, d\mathbf{x} \Big/ \int_\Omega \bar{\omega}(a-\bar{\omega})(\nabla\psi)^2 d\mathbf{x}. \qquad (11)$$

The growth of entropy (10) implies that $A > 0$, and we finally obtain [11] our evolution equations for $\bar{\omega}$:

$$\bar{\omega}_t + \nabla\cdot(\bar{\omega}\mathbf{u}) = A\nabla^2\bar{\omega} + A\beta\nabla\cdot[(a-\bar{\omega})\bar{\omega}\nabla\psi],$$
$$[\nabla\bar{\omega} + \beta(a-\bar{\omega})\bar{\omega}\nabla\psi]\cdot\mathbf{n} = 0, \text{ on } \partial\Omega. \qquad (12)$$

This is similar to the Navier-Stokes equation but the usual diffusion term is corrected to exactly satisfy the energy conservation, while ensuring a monotonous increase of entropy. This correction is nonlocal in terms of the vorticity, which is not surprising since the vorticity variations influence velocity over long range.

The same method could be used for the case of n vorticity levels, but we are not able to determine the corresponding matrix of diffusion coefficients by pure symmetry arguments. We instead reformulate the problem and propose a variational principle to determine the fluxes $\mathbf{J}_i$. We will make the following hypothesis:

(H1) For any given state $\mathbf{p} = (p_1, \ldots, p_n)$, the rate of entropy production $\dot{S}$ when the system is at $\mathbf{p}$ is a functional of $\mathbf{p}$; we will write $\dot{S}=C\{\mathbf{p}\}$.

(H2) The system distributes its fluxes in order to minimize the diffusion energy. That is, the actual fluxes $\mathbf{J}_i^*$ will satisfy the following variational problem (V.P.′):

$$E_d\{J_1^*,\ldots,J_n^*\}=\min E_d\{J_1,\ldots,J_n\},$$

under the constraints

$$\text{(C1) } \dot{S}=C\{\mathbf{p}\}, \quad \text{(C2) } \sum_i \mathbf{J}_i=0, \quad \text{(C3) } \dot{E}=0.$$

This variational problem always has a unique solution $\mathbf{J}_1^*,\ldots,\mathbf{J}_n^*$. There are two Lagrange multipliers A,B such that for any variations $\delta\mathbf{J}_i$ satisfying $\sum\delta\mathbf{J}_i=0$, we have $\delta E_d=A\,\delta\dot{S}+B\,\delta\dot{E}$. A straightforward calculation then gives that $\mathbf{J}_i^*/p_i-A\nabla s_i-Ba_i\nabla\psi=F(\mathbf{x})$, for all i. The function $F(\mathbf{x})$ is calculated thanks to the relation $\sum\mathbf{J}_i^*=0$, and we get

$$\mathbf{J}_i^*=-A\nabla p_i-B(\bar{\omega}-a_i)p_i\nabla\psi. \tag{13}$$

The constants A,B can be now calculated thanks to the constraints (C1) and (C3). The conservation of energy (C3) gives the ratio $\beta=B/A$,

$$\beta=-\int_\Omega\nabla\psi\cdot\nabla\bar{\omega}\,d\mathbf{x}\Big/\int_\Omega\left[\sum a_i^2p_i-\bar{\omega}^2\right](\nabla\psi)^2d\mathbf{x}. \tag{14}$$

From the strict convexity of the function $a\to a^2$, it follows that $(\sum a_ip_i)^2\le\sum a_i^2p_i$, with equality only when all the p_i are equal to 0 or 1. It follows that β is defined for any mixed state. The entropy condition (C1) then relates A to the entropy variation $C\{\mathbf{p}\}$ and gives that A must be of the same (positive) sign as $C\{\mathbf{p}\}$. Finally, let us summarize our relaxation equations for an n-level vorticity:

$$(p_i)_t+\nabla\cdot(p_i\mathbf{u}+\mathbf{J}_i^*)=0,\quad i=1,\ldots,n,$$
$$\mathbf{J}_i^*=-A\{\mathbf{p}\}[\nabla p_i-\beta(\bar{\omega}-a_i)p_i\nabla\psi],\ \text{with } \mathbf{J}_i^*\cdot\mathbf{n}=0 \text{ on } \partial\Omega,$$
$$\mathbf{u}=\nabla\times(\psi\mathbf{k}),\quad \bar{\omega}=\sum_i a_ip_i=-\nabla^2\psi,\quad \text{with } \psi=0 \text{ on } \partial\Omega. \tag{15}$$

In the particular case of a vortex patch with vorticity a surrounded by irrotational fluid, (15) and (14) reduce to our previous results (12) and (11). Notice that we do not need to make an *a priori* hypothesis of local equilibrium or of linear dependence of the fluxes on the forces. The linearity is merely a consequence of our variational principle [12]. In the general case of a continuous initial vorticity distribution, (15) is straightforwardly generalized by replacing the finite set $p_i(t,\mathbf{x})$ by the continuous vorticity distribution $p(a,t,\mathbf{x})$.

In the particular case $\Omega=B(O,R)$, we must also take into account the constraint given by the angular momentum. Then the fluxes have to satisfy the supplementary condition $\int_\Omega\mathbf{x}\cdot\mathbf{J}_\omega d\mathbf{x}=0$. Thus the optimal fluxes $\mathbf{J}_i^*$ can be written

$$\mathbf{J}_i^*=-A[\nabla p_i-(\bar{\omega}-a_i)p_i(\beta\nabla\psi+\gamma\mathbf{x})], \tag{16}$$

where β and γ are determined by the linear system

$$\beta\int\theta^2\nabla\psi^2d\mathbf{x}+\gamma\int\theta^2\mathbf{x}\cdot\nabla\psi\,d\mathbf{x}=-\int\nabla\psi\cdot\nabla\bar{\omega}\,d\mathbf{x},$$
$$\beta\int\theta^2\mathbf{x}\cdot\nabla\psi\,d\mathbf{x}+\gamma\int\theta^2\mathbf{x}^2d\mathbf{x}=-\int\mathbf{x}\cdot\nabla\bar{\omega}\,d\mathbf{x}, \tag{17}$$

where $\theta^2=\sum_i a_i^2p_i-\bar{\omega}^2$. We easily show, using the Schwarz inequality, that the determinant of this system is always >0 (we suppose that some mixing has occurred, so that $\theta^2>0$ everywhere), and the solution for β and γ is therefore unique, except in the degenerate case of a solid body rotation. In the case of a channel with periodic boundary conditions, the linear momentum is conserved instead of the angular momentum [6], and this case can be treated similarly.

The link with the equilibrium theory is confirmed by the following proposition: Let us suppose that the solution $p_i(t,\mathbf{x})$, $i=1,\ldots,n$, of the system (15) converges (in a strong enough sense) when t goes to infinity, towards a stationary state $p_i(\mathbf{x})$, such that $p_i(\mathbf{x})\ge\varepsilon>0$, for $i=1,\ldots,n$. Then $p_i(\mathbf{x})$ is [13] a Gibbs state (4) with $\beta=\lim\beta(t)$, $t\to\infty$. To prove this proposition, we write from (10) to (13)

$$\dot{S}=-\int_\Omega\sum_i\frac{1}{p_i}[\nabla p_i-\beta(\bar{\omega}-a_i)p_i\nabla\psi]\cdot\mathbf{J}_i^*\,d\mathbf{x}$$
$$-\int_\Omega\sum_i\beta(\bar{\omega}-a_i)\nabla\psi\cdot\mathbf{J}_i^*\,d\mathbf{x},$$

but this last term is zero, due to the fact that $\sum_i\mathbf{J}_i^*=0$ and $\int_\Omega\nabla\psi\cdot\mathbf{J}_\omega^*\,d\mathbf{x}=0$.

Thus it follows that

$$\dot{S}=\int_\Omega\sum_i\frac{A}{p_i}[\nabla p_i-\beta(\bar{\omega}-a_i)p_i\nabla\psi]^2d\mathbf{x}.$$

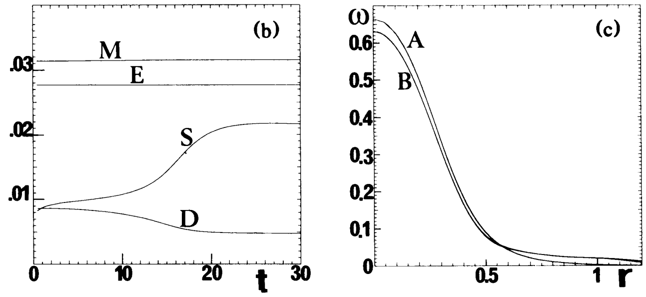

FIG. 1. The vortex merging obtained by our model equations (in a square domain with side 3.14, resolution 64^2 grid points, $A=10^{-2}$). The two initial spots have a vorticity $a=1$, with smoothed edges to avoid initial vorticity discontinuities. (a) Vorticity contours with interval 0.1. (b) Time evolution of the angular momentum (M), energy (E), entropy (S), enstrophy (D). (c) Comparison of the final radial vorticity profile (A) with the result from Navier-Stokes equations with the same initial condition (B, resolution 256^2, viscosity 3×10^{-5}, time $t=400$).

In the limit of a steady state, $\dot{S}$ vanishes so that each current $\mathbf{J}_i^*$ vanishes, and by subtracting two terms we get $\nabla \ln(p_i/p_n)+\beta(a_i-a_n)\nabla\psi=0$, $i=1,\ldots,n-1$. It implies that $\ln(p_i/p_n)+\beta(a_i-a_n)\psi$ has a constant value $-\alpha_i$ on Ω. Then, using $\sum p_i(\mathbf{x})=1$, we get indeed the Gibbs state (4).

As an example of an application, we compute the merging of two vortices (Fig. 1). We use a pseudospectral scheme, and the computation domain is square for convenience (with periodic boundary conditions in a twice wider domain obtained by symmetry with respect to the wall). However, in order to simulate an infinite domain, we also impose the conservation of angular momentum. Therefore we solve (6) with a single nonzero vorticity level a, with diffusion current (16) and condition (17). The energy and angular momentum are indeed constant, while entropy is increasing and enstrophy $\int_\Omega \omega^2 d\mathbf{x}$ is decreasing [Fig. 1(b)]. The viscosity $A=10^{-2}$ results from an adjustment: With much lower values (0.2×10^{-2}), vorticity structures at the grid scale are not sufficiently damped and numerical errors appear (in particular entropy fluctuates). For high values of A, the vorticity diffusion is too strong, and the merging occurs too quickly. Nevertheless the final state, a steady axisymmetric vortex, is independent of A: It is the equilibrium state corresponding to the initial energy and angular momentum. Its radial vorticity profile is found in good agreement with a direct computation of the Navier-Stokes equations at high resolution and weak viscosity [Fig. 1(c)].

In conclusion, we have proposed a set of evolution equations which provides a convenient algorithm to compute the equilibrium states corresponding to a given initial datum. Besides this fact, these equations can be considered as a small-scale modeling of the Euler flow. They have the property of smoothing the vorticity fluctuations at small scales, but unlike ordinary viscosity, they conserve all the constants of the motion of the Euler system and can drive the system towards nontrivial organized structures at long times. The irreversibility of this model is expressed by the monotonous increase of the mixing entropy. The smoothing effect can be adjusted by the viscosity coefficient A, according to the spatial resolution of the explicit scales. Of course we have made strong assumptions on the diffusion process, and this is only a first step towards a more sophisticated modeling of the Euler flows.

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[10] When Ω is the disk $B(O,R)$ we must also take into account the angular momentum. This supplementary constraint introduces a new Lagrange multiplier γ, and $Z(\psi)=\sum_i \exp\{-\alpha_i-a_i[\beta\psi+\frac{1}{2}\gamma(R^2-\mathbf{x}^2)]\}$.
[11] From an initial vorticity ω_0 which is not mixed (i.e., $\omega_0=0$ or a everywhere), the denominator in (11) is zero and β is not defined at $t=0$. However, since the relevant term in (12) is also proportional to $\bar{\omega}(a-\bar{\omega})$, we can suppress this factor in both (11) and (12) at the initial time and come back to the actual equation after some diffusion has occurred.
[12] One can easily check that our variational principle is equivalent to the following formulation, which is clearly in the spirit of Jaynes's ideas [E. T. Jaynes, in *The Minimum Entropy Production Principle,* edited by R. D. Rosenkrantz, Pallas Paperback Series (Kluwer Academic, Dordrecht, 1989)]. For any given state $\mathbf{p}$, the diffusion energy E_d when the system is at $\mathbf{p}$ is a functional of $\mathbf{p}$: $E_d=C\{\mathbf{p}\}$. The system distributes its fluxes in order to maximize the rate of entropy production. That is the actual fluxes $\mathbf{J}_i^*$ will satisfy $\dot{S}\{\mathbf{J}_1^*,\ldots,\mathbf{J}_n^*\}=\max\dot{S}\{\mathbf{J}_1,\ldots,\mathbf{J}_n\}$ under the constraints $E_d=C\{\mathbf{p}\}$. (C2) and (C3) are unchanged. This actually shows why the system is likely to reach a maximum-entropy state.
[13] If in addition we suppose that $\beta>\beta_c=-\lambda_1/\sup\{a_i^2; i=1,\ldots,n\}$, where λ_1 is the first eigenvalue of the Laplacian in the domain Ω, then $(p_1,\ldots,p_n)$ is the unique maximum of the entropy functional under the constraints given by the constants of the motion, as shown in Refs. [3,4]. Then the system can only reach this unique optimal state.