Wiggly Relativistic Strings

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We derive the equations of motion for general strings, i.e., strings with arbitrary relation between tension τ and energy per unit length ϵ. The renormalization of τ and ϵ that results from averaging out small scale wiggles on the string is obtained in the general case to lowest order in the amount of wiggliness. For Nambu-Goto strings we find deviations from the equation of state $\epsilon\tau$ = const in higher orders. Finally we argue that wiggliness may radically modify the cosmic gauge string scenario.

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Computer simulations [1] have shown that cosmic gauge strings acquire much more small scale structure than had been thought previously [2]. The effect of small scale structure is, generally speaking, to increase the energy per unit length of a gauge string and to decrease its tension when a coarse grained description of the wiggly string is adopted [3, 4]. While this fact has been appreciated in the past, it appears that there is no general treatment yet of the renormalization of string tension and energy per unit length due to small scale wiggliness. It appears even that there is no relativistic description in place for the motion of strings when the tension differs from the energy per unit length. The present paper aims to fill this gap. The formalism developed below can be applied to all strings and we hope it will prove useful in other contexts. An example which comes to mind is a gauge string with fermion zero modes attached to it [5]. When a large number of particles are attached to a string, they act collectively as a fluid adding to the energy per unit length but subtracting from the tension.

Consider then a general string, i.e., an object whose stress-energy–momentum tensor is localized on a line in space. Let $X^\mu(\sigma)$ be the location of the string world sheet with respect to a Lorentz reference frame. $\sigma=(\sigma^0,\sigma^1)$ are arbitrarily chosen coordinates parametrizing points on the world sheet. The associated two-dimensional metric is, as usual,

$$h_{ab}(\sigma)=\partial_a X^\mu \partial_b X_\mu\ , \quad a,b=0,1\ . \tag{1}$$

At point $X^\mu(\sigma)$ on the string lives a two-dimensional stress-energy–momentum tensor

$$t^{ab}=(\epsilon-\tau)u^a u^b+\tau h^{ab}, \tag{2}$$

where $\epsilon(\sigma)$ is the energy per unit length of the string, $\tau(\sigma)$ its tension, and $u^a(\sigma)$ is the fluid velocity parallel to the string; $u^a u_a=+1$. The four-dimensional stress-energy–momentum tensor is

$$T^{\mu\nu}(x)=\int d^2\sigma\sqrt{-h}\ t^{ab}(\sigma)\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma)\delta^4(x-X(\sigma))\ , \tag{3}$$

where $h=\det(h_{ab})$. This expression is both invariant under two-dimensional reparametrizations and covariant under four-dimensional Lorentz transformations. Its validity in the neighborhood of each point on the world sheet can be verified explicitly by choosing a Lorentz frame which is instantaneously at rest with respect to the string at that point. The motion of the string must be such that $\partial_\mu T^{\mu\nu}(x)=0$. It is easy to show that this condition is equivalent to

$$\partial_a[\sqrt{-h}\ t^{ab}(\sigma)\partial_b X^\mu]=0 \quad (\mu=0,1,2,3)\ . \tag{4}$$

To provide a complete description of the string dynamics, Eq. (4) must be supplemented by an equation of state:

$$\tau=\tau(\epsilon)\ . \tag{5}$$

We then have five equations for the five unknowns $\mathbf{X}_\perp(\sigma)$, $\epsilon(\sigma)$, $\tau(\sigma)$, and $\beta(\sigma)\equiv u^1(\sigma)/u^0(\sigma)$. $\mathbf{X}_\perp(\sigma)$ represents the two transverse degrees of freedom of the string. $\beta(\sigma)$ is its longitudinal velocity. Equations (4) and (5) uniquely specify the motion of the string that results from arbitrary initial conditions. Note that our description is Lorentz invariant as well as generally covariant in the two-dimensional sense.

The case of the Nambu-Goto (NG) string is included in this description. The well-known NG equations of motion are Eq. (4) with $t^{ab}(\sigma)$ replaced by $h^{ab}(\sigma)$. Let us first note that two of these equations are merely mathematical identities since

$$\partial_e X_\mu\, \partial_a[\sqrt{-h}\ h^{ab}(\sigma)\partial_b X^\mu]=0 \quad (e=0,1) \tag{6}$$

follows from Eq. (1) and nothing else. The other two NG equations specify the motion of $\mathbf{X}_\perp(\sigma)$. Now, returning to our description of a general string, let us adopt the equation of state $\tau=\epsilon$. It is easy to show, using Eqs. (6), that two of the Eqs. (4) are equivalent to

$$\partial_a\tau(\sigma)=0, \quad a=0,1\ . \tag{7}$$

They therefore imply that τ must be a constant. The two remaining Eqs. (4) are the nontrivial NG equations that

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determine the motion of $\mathbf{X}_\perp(\sigma)$. Thus we have learned that $\tau = \text{const}$ is the only consistent way to have $\tau = \epsilon$.

The equation of state appropriate to a particular kind of string must be derived from the relevant microphysics. As in other studies of fluid dynamics, an average over a large number of microconfigurations consistent with a given macroscopic description must be performed to obtain the energy per unit length ϵ and the tension τ of the string. The equation of state gives the relationship between ϵ and τ when the string is slowly stretched (i.e., the stretching time scale is long compared to the microphysics time scales). The focus of our paper is the renormalization of ϵ and τ due to small scale wiggles on a string with arbitrary equation of state. We must define an averaging length scale $\lambda \equiv \frac{2\pi}{k}$. $\epsilon(k)$ and $\tau(k)$ are the values of the energy per unit length and tension of the string when all wiggles of wavelength shorter than λ are averaged over. We will derive the renormalization group equations for ϵ and τ due to transverse and longitudinal wiggles to second order in the amplitude of the wiggles.

Consider a string which lies on average along the x axis. We choose t and x as world-sheet coordinates. Thus $X^\mu(\sigma) = (t, x, y(t,x), z(t,x))$. In this gauge, Eqs. (4) are

$$\partial_a(\sqrt{-h}\, t^{ab}) = 0\,, \tag{8a}$$
$$t^{ab}\partial_a\partial_b y = t^{ab}\partial_a\partial_b z = 0\,. \tag{8b}$$

Let us first discuss transverse wiggles. Consider a string of equation of state $\tau = \tau(\epsilon)$ stretched along the x axis. At rest ($\beta = y = z = 0$) the string has energy per unit length ϵ_0 and tension $\tau_0 = \tau(\epsilon_0)$. At time $t = 0$, the string is given a transverse velocity in the y direction: $\dot{y}(0,x) = \beta_T \sin kx$. We assume $\beta_T \ll 1$ and expand Eqs. (8) in powers of β_T. This yields to lowest order

$$\epsilon_0 \ddot{y} - \tau_0 y'' = 0, \tag{9a}$$
$$\ddot{\epsilon}_1 - v_L^2 \epsilon_1'' = (-\epsilon_0\partial_t^2 + \tau_0\partial_x^2)\tfrac{1}{2}(\dot{y}^2 + y'^2), \tag{9b}$$
$$(\epsilon_0 - \tau_0)\dot{\beta} = \tau_0 y'(\ddot{y} - y'') + v_L^2 \epsilon_1'\,, \tag{9c}$$

for the time evolution of y, β, and $\epsilon_1 = \epsilon - \epsilon_0$. By definition $v_L^2 \equiv -\frac{d\tau}{d\epsilon}|_0$. As usual, dots and primes denote derivatives with respect to t and x. Equation (9a) implies $y = (\beta_T/w)\sin kx \sin \omega t$ with $\omega = k v_T$, where $v_T = (\frac{\tau_0}{\epsilon_0})^{1/2}$ is the phase velocity of transverse wiggles. Equations (9b) and (9c) determine $\epsilon_1(\sigma)$ and $\beta(\sigma)$, both of which are of order $\dot{y}^2$. To obtain the renormalized values $\bar{\epsilon}_T$ and $\bar{\tau}_T$ of the energy per unit length and tension, we calculate $\langle T_{\mu\nu}(x)\rangle$ to second order in $\dot{y}$. The result, including the contribution from wiggles in the x-z plane, is

$$\bar{\epsilon}_T = \epsilon_0 + \epsilon_0(\langle \dot{y}^2\rangle + \langle \dot{z}^2\rangle), \tag{10a}$$
$$\bar{\tau}_T = \tau_0 - \frac{1}{2}(\langle \dot{y}^2\rangle + \langle \dot{z}^2\rangle)\left[\tau_0 + \epsilon_0 + v_L^2\epsilon_0\left(1 - \frac{\epsilon_0}{\tau_0}\right)\right]. \tag{10b}$$

Next, let us discuss longitudinal wiggles. Again we start with a string which is at rest, stretched along the x axis. In this state, it has energy per unit length ϵ_0 and tension $\tau_0 = \tau(\epsilon_0)$. At time $t = 0$, the string is given a longitudinal velocity $\beta(0,x) = \beta_L \sin kx$. Provided $\beta_L \ll 1$, Eqs. (9) are still valid but now $y = z = 0$. They imply that $\epsilon_1 \sim (\epsilon_0 - \tau_0)v_L^{-1}\beta_L$ and that $v_L = (-\frac{d\tau}{d\epsilon}|_0)^{1/2}$ is the phase velocity of longitudinal wiggles. For the renormalization of ϵ and τ due to longitudinal wiggles we find, up to second order in β:

$$\bar{\epsilon}_L = \epsilon_0 + 2(\epsilon_0 - \tau_0)\langle\beta^2\rangle, \tag{11a}$$
$$\bar{\tau}_L = \tau_0 - (\epsilon_0 - \tau_0)\langle\beta^2\rangle\left[1 + v_L^2 + (\epsilon_0 - \tau_0)\frac{d\ln v_L}{d\epsilon}\bigg|_0\right]. \tag{11b}$$

Thus the promised renormalization group equations for $\epsilon(k)$ and $\tau(k)$ are

$$-\frac{d\epsilon}{d\ln k} = W_T(k)\epsilon + 2W_L(k)(\epsilon - \tau), \tag{12a}$$

$$-\frac{d\tau}{d\ln k} = -\frac{1}{2}W_T(k)\left[\tau + \epsilon + v_L^2\epsilon\left(1 - \frac{\epsilon}{\tau}\right)\right] - W_L(k)(\epsilon - \tau)\left[1 + v_L^2 + (\epsilon - \tau)\frac{d\ln v_L}{d\epsilon}\right], \tag{12b}$$

where $W_T(k)$ and $W_L(k)$ are the spectral densities on a $\ln k$ scale of respectively $\langle \dot{y}^2\rangle + \langle \dot{z}^2\rangle$ and $\langle\beta^2\rangle$. Equations (12) relate the values of ϵ and τ at one scale to their values at a vastly different scale provided that $W_L, W_T \ll 1$ at all intermediate scales. Note that we have not yet obtained how the equation of state itself changes from scale to scale. To do so we need to analyze the response of the wiggles to adiabatic stretching of the string. We leave this to a future publication which will also contain the details of the calculation that led to Eqs. (10)–(12).

Let us consider the case of wiggly Nambu-Goto strings. At the shortest distance scale k_0 we have $\epsilon = \tau = \mu$, where μ is the bare string tension. At slightly longer distance scales Eqs. (12) imply $\epsilon(k) = \mu[1 + W_T(k_0)\ln k_0/k]$ and $\tau(k) = \mu[1 - W_T(k_0)\ln k_0/k]$. Therefore in the neighborhood of $k = k_0$ we have the equation of state $\epsilon\tau = \mu^2$. Moreover, a short calculation shows that Eqs. (12) imply $\frac{d(\epsilon\tau)}{d\ln k} = 0$ when the equation of state is $\epsilon\tau = \text{const}$. This equation of state is therefore a fixed point of the renormalization group equations (12). Thus in the particular case of wiggly Nambu-Goto strings, Eqs. (12) do by themselves establish the equation of state to be $\epsilon\tau = \mu^2$ in lowest order. (As was already emphasized, for a general string the renormalization group equations for ϵ and τ do not by themselves provide enough information to determine the equation of state. The Nambu-Goto string is an exception in this regard.) The equation of state

$\epsilon\tau = \mu^2$ was found earlier by Carter [3] and Vilenkin [4]. It seems that we have found considerable support for it. However, we will now show that in general $\epsilon\tau \neq \mu^2$ for wiggly Nambu-Goto strings although $\epsilon\tau = \mu^2$ may be an excellent approximation in many cases. The fact that $\epsilon\tau \neq \mu^2$ in general does not contradict what we have said so far because Eqs. (12) are valid only in lowest order.

It is well known that an arbitrary motion of a Nambu-Goto string [6] is given by $\mathbf{x}(t,\sigma) = \frac{1}{2}[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)]$, where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary functions subject only to the constraint $\mathbf{a}'^2 = \mathbf{b}'^2 = 1$. To describe a wiggly NG string lying on average along the x axis we write

$$\begin{aligned} \mathbf{a}(\sigma - t) &= [(\sigma - t)\gamma_1 + f_1(\sigma - t),\ g_{1y}(\sigma - t),\ g_{1z}(\sigma - t)], \\ \mathbf{b}(\sigma + t) &= [(\sigma + t)\gamma_2 + f_2(\sigma + t),\ g_{2y}(\sigma + t),\ g_{2z}(\sigma + t)], \end{aligned} \tag{13}$$

where γ_1 and γ_2 are constants and f_1, f_2, g_{1y}, g_{1z}, g_{2y}, and g_{2z} are functions which average to zero and which describe the wiggles on the string. These functions are not all independent since they must obey the gauge condition $\mathbf{a}'^2 = \mathbf{b}'^2 = 1$. Let us choose (t, x) as the world-sheet coordinates of the averaged string. It is easy to show that in these coordinates the two-dimensional stress-energy–momentum tensor of the averaged string is given by

$$\begin{aligned} t^{00} &= \mu\left\langle \frac{1}{\mid \frac{dx}{d\sigma} \mid} \right\rangle = \mu\left\langle \frac{2}{\gamma_1 + \gamma_2 + f_1'(\sigma - t) + f_2'(\sigma + t)} \right\rangle, \quad t^{01} = \mu\left\langle \frac{-\gamma_1 + \gamma_2 - f_1'(\sigma - t) + f_2'(\sigma + t)}{\gamma_1 + \gamma_2 + f_1'(\sigma - t) + f_2'(\sigma + t)} \right\rangle, \\ t^{11} &= -2\mu\left\langle \frac{[\gamma_1 + f_1'(\sigma - t)][\gamma_2 + f_2'(\sigma + t)]}{\gamma_1 + \gamma_2 + f_1'(\sigma - t) + f_2'(\sigma + t)} \right\rangle . \end{aligned} \tag{14}$$

Clearly $\epsilon\tau = -t^{00}t^{11} + (t^{01})^2 \neq \mu^2$ in general. For example, consider the particular case where the wiggles are reflection symmetric on average ($\gamma_1 = \gamma_2 \equiv \gamma$, $\langle f_1'^p \rangle = \langle f_2'^p \rangle \equiv \langle f'^p \rangle$ for $p = 2, 3, \ldots$). Then an expansion of t^{ab} in powers of f' yields

$$\begin{aligned} \epsilon = t^{00} &= \frac{\mu}{\gamma}\left[1 + \frac{1}{2\gamma^2}\langle f'^2 \rangle - \frac{1}{4\gamma^3}\langle f'^3 \rangle + \frac{1}{8\gamma^4}(\langle f'^4 \rangle + 3\langle f'^2 \rangle^2) - \cdots \right], \\ \tau = -t^{11} &= \gamma\mu\left[1 - \frac{1}{2\gamma^2}\langle f'^2 \rangle + \frac{1}{4\gamma^3}\langle f'^3 \rangle - \frac{1}{8\gamma^4}(\langle f'^4 \rangle - \langle f'^2 \rangle^2) + \cdots \right]. \end{aligned} \tag{15}$$

Equations (15) show that $\epsilon\tau \neq \mu^2$ although the second- and third-order terms in the expansion of $\epsilon\tau - \mu^2$ vanish. That the second-order term vanishes, we already knew from Eqs. (12). In general, one has

$$\frac{\epsilon\tau}{\mu^2} = 1 + \frac{1}{4\gamma^2}\langle f_1'^2 \rangle \langle f_2'^2 \rangle + \cdots . \tag{16}$$

Note that if the wiggles are purely transverse ($f_1' = f_2' = 0$) then $\epsilon\tau = \mu^2$ to all orders [4]. However, it is clear that one must allow $f_1', f_2' \neq 0$. Physically this corresponds to the possibility of longitudinal wiggles once the Nambu-Goto string has $\epsilon > \tau$ because of transverse wiggles.

Finally, we would like to speculate on the behavior of cosmic gauge strings. Let us assume that higher-order terms in the renormalization group equations (12) do not play an important role. The equation of state is then $\epsilon\tau = \mu^2$, and

$$-\frac{d\epsilon}{d\ln k} = W_T\epsilon + 2W_L\left(\epsilon - \frac{\mu^2}{\epsilon}\right) . \tag{17}$$

How large are W_T and W_L? At cosmic time t, when the correlation length of the string network is $\xi(t)$, the strings carry wiggles which have been inherited from earlier times when the correlation length was shorter. For $\lambda = \frac{2\pi}{k}$ somewhat shorter than $\xi(t)$, wiggles are abundant and the corresponding values of W_T and W_L are large, of order 1. For $\lambda < G\mu t$, W_T and W_L are exponentially suppressed because the decay time of wiggles on the bare string into gravitational radiation is of order $(G\mu)^{-1}\lambda$ [7]. For $G\mu t < \lambda < \xi(t)$, the values of W_T and W_L are the outcome of a number of competing processes [8] some of which tend to increase and some of which tend to decrease the size of wiggles associated with the corresponding length scales. Stretching of the strings (e.g., by Hubble expansion) and the production of loops by self-intersection with reconnection tend to decrease the size of wiggles, whereas shortening of the string after reconnections have occurred and the production of kinks, also as a result of reconnections, tend to increase the size of wiggles. We will assume here that W_T and W_L have approximately constant values for all λ: $G\mu t < \lambda < \xi(t)$. We make this assumption not because we believe that it is necessarily correct but as a means to explore the effect of small scale wiggliness on the cosmic gauge string scenario. With regard to the computer simulations, it is unclear to us whether they are in disagreement with this assumption. There is so far no published result describing unambiguously the spectrum of wiggliness.

At any rate, if W_T and W_L are approximately constant for $G\mu t < \lambda < \xi(t)$, then, from Eq. (17),

$$\epsilon = \epsilon(\xi) = \frac{\mu^2}{\tau} \sim \mu\left(\frac{\xi(t)}{G\mu t}\right)^{W_T+2W_L} . \tag{18}$$

The typical velocity of cosmic strings is then

$$v(t) = \left[\frac{\tau(t)}{\epsilon(t)}\right]^{\frac{1}{2}} \sim \left(\frac{G\mu t}{\xi(t)}\right)^{W_T+2W_L} . \tag{19}$$

One expects the correlation length to be $\xi(t) = v(t)t$. This determines $\xi(t) \sim t(G\mu)^\alpha$, where $\alpha = \frac{W_T+2W_L}{W_T+2W_L+1}$. The density of strings today is then given by

$$\Omega_{\rm str} \sim (6\pi G t^2)\frac{\epsilon}{\xi^2(t)} \sim (6\pi)(G\mu)^{1-3\alpha} . \tag{20}$$

Because of limits on the anisotropy of the microwave background radiation, $\Omega_{\rm str}$ must be much smaller than 1. This requires $\alpha < \frac{1}{3}$ or $W_T + 2W_L < 0.5$. Even if this condition is satisfied, the limit on $G\mu$ may be much more severe than it is in the usual scenario which assumes $\alpha = 0$. For example, if $W_T + 2W_L = 0.2$, $\Omega_{\rm str} < 10^{-5}$ implies $G\mu < 10^{-12}$ instead of $G\mu < 10^{-6}$ in the usual scenario.

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