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Integrability and the Motion of Curves

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Recently discovered connections between integrable evolution equations and the motion of curves are based on the following fact: The Serret-Frenet equations are equivalent to the Ablowitz-Kaup-Newell-Segur (AKNS) scattering problem at zero eigenvalue. This equivalence identifies those evolution equations, integrable or not, that can describe the motion of curves.

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A variety of physical processes can be modeled in terms of the motion of curves, including the dynamics of vortex filaments in fluid dynamics [1], the growth of dendritic crystals in a plane [2], and more generally, the planar motion of interfaces [3]. In an intriguing recent paper, Goldstein and Petrich [4] related integrable evolution equations from the modified Korteweg–de Vries (mKdV) hierarchy to motions of closed curves in a plane. Being integrable, these motions conserve infinitely many global invariants, including both the total length of the curve and (if the curve is closed) its enclosed area. Their work brings to mind earlier works of Hasimoto [1] and Lamb [5], who showed that the nonlinear Schrödinger (NLS) equation describes a family of motions of curves in 3-space. Because NLS belongs to the same hierarchy of integrable equations as mKdV, all of these works raise the same question: *What is the relationship between integrable evolution equations and the motion of curves, either in the plane or in 3-space?* The purpose of this paper is to answer this question.

The first step is to describe the motion of curves. Consider a smooth curve in 3-space, parametrized by α. Let $\boldsymbol{r}(\alpha,t)$ denote the position vector of a point on the curve at time t. There is a metric on the curve,

$$g(\alpha,t) = (\partial \boldsymbol{r}/\partial\alpha)\partial\boldsymbol{r}/\partial\alpha; \tag{1}$$

the arclength along the curve is given by

$$s(\alpha,t) = \int_0^\alpha \sqrt{g(\alpha',t)}\, d\alpha', \tag{2}$$

and we may use either $\{\alpha,t\}$ or $\{s,t\}$ as coordinates of a point on the curve. At $\boldsymbol{r}(\alpha,t)$, let $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ denote respectively the unit tangent, normal, and binormal vectors, defined in the usual way (i.e., $\boldsymbol{t} := \partial\boldsymbol{r}/\partial s = g^{-1/2}\,\partial\boldsymbol{r}/\partial\alpha$, etc.). These vectors satisfy the familiar Serret-Frenet equations [6],

$$\frac{\partial \boldsymbol{t}}{\partial s} = \kappa\boldsymbol{n}, \quad \frac{\partial \boldsymbol{n}}{\partial s} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \quad \frac{\partial \boldsymbol{b}}{\partial s} = -\tau\boldsymbol{n}, \tag{3}$$

where $\partial(\ \ )/\partial s := \partial(\ \ )/\partial s|_t$, and $\kappa(s,t)$ and $\tau(s,t)$ are respectively the curvature and torsion of the curve at $\boldsymbol{r}$. (Note that the sign of κ here differs from that in Ref. [4].) Motion of a point on the curve can be specified in the form

$$\dot{\boldsymbol{r}} := \left.\frac{\partial\boldsymbol{r}}{\partial t}\right|_\alpha = U\boldsymbol{n} + V\boldsymbol{b} + W\boldsymbol{t}, \tag{4}$$

and the motion is said to be *local* if $\{U,V,W\}$ depend only on local values of $\{\kappa,\tau\}$ and their s derivatives [2, 4].

Two-dimensional motion.—Motion in a plane occurs if $V \equiv 0$ and $\tau \equiv 0$ in (3) and (4). Then the evolution in time of the other variables is determined by requiring $\frac{\partial}{\partial t}\frac{\partial}{\partial\alpha}\boldsymbol{r}(\alpha,t) = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial t}\boldsymbol{r}(\alpha,t)$, and using $\partial(\ \ )/\partial\alpha|_t = g^{1/2}\partial(\ \ )/\partial s|_t$. For 2D motion, the result is

$$\dot{\boldsymbol{t}} = \left(\frac{\partial U}{\partial s} + \kappa W\right)\boldsymbol{n}, \quad \dot{\boldsymbol{n}} = -\left(\frac{\partial U}{\partial s} + \kappa W\right)\boldsymbol{t},$$
$$\dot{g} = 2g\left(\frac{\partial W}{\partial s} - \kappa U\right), \quad \dot{\kappa} = \left(\frac{\partial^2 U}{\partial s^2} + \kappa^2 U + \frac{\partial\kappa}{\partial s}W\right), \tag{5}$$

from which it follows that

$$\dot{s} = \int_0^\alpha g^{1/2}\left(\frac{\partial W}{\partial s} - \kappa U\right) d\alpha'$$

or

$$\dot{s}(\alpha,t) = W(s,t) - \int_0^s \kappa U\, ds', \tag{6}$$

provided $W(0,t)=0$. Because $\dot{\kappa}(s,t) = \partial\kappa/\partial t + \dot{s}\,\partial\kappa/\partial s$, it follows that [4]

$$\frac{\partial\kappa}{\partial t} = \left(\frac{\partial^2 U}{\partial s^2} + \kappa^2 U + \frac{\partial\kappa}{\partial s}\int^s \kappa U\, ds'\right) =: \Omega U. \tag{7}$$

Notice that $\kappa(s,t)$, and hence the 2D motion of the curve, follows from specifying $U(s,t)$ and then integrating (7). $W(s,t)$ determines how points parametrized by α move along the curve, but it does not affect the shape of the curve.

The next step is to introduce integrability. A standard description of a class of integrable evolution equations, including the mKdV hierarchy, was given in Ref. [7]. The basic scattering problem for $\boldsymbol{v}(s,t;\zeta)$ is

$$\begin{pmatrix} \dfrac{\partial v_1}{\partial s} \\ \dfrac{\partial v_2}{\partial s} \end{pmatrix} = \begin{pmatrix} i\zeta & q(s,t) \\ r(s,t) & -i\zeta \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{8}$$

Integrable evolution equations for $q(s,t)$ and $r(s,t)$ are obtained by (i) specifying time dependence for $\boldsymbol{v}$, and (ii) requiring compatibility (i.e., $\frac{\partial}{\partial s}\frac{\partial}{\partial t}\boldsymbol{v} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\boldsymbol{v}$) for all $\{s,t,\zeta\}$.

The relation between these two subjects is as follows. For curves in a plane, $\boldsymbol{t}$ and $\boldsymbol{n}$ each have two components, related by (3) according to

$$\frac{\partial t_j}{\partial s} = \kappa n_j, \quad \frac{\partial n_j}{\partial s} = -\kappa t_j, \quad j = 1,2. \tag{9}$$

These are equivalent to (8) at $\zeta = 0$, with $q = \kappa$, $r = -q$. Consequently, any integrable evolution equation from the mKdV hierarchy with $\{r = -q\}$ is compatible with (7). For example, if we choose $U = -\partial\kappa/\partial s$, then $\kappa(s,t)$ evolves according to the focusing version of mKdV [8],

$$\kappa_t + (3/2)\kappa^2\kappa_s + \kappa_{sss} = 0. \tag{10}$$

No choice of U would permit κ to satisfy the defocusing mKdV, $\kappa_t - (3/2)\kappa^2\kappa_s + \kappa_{sss} = 0$, because this would require $\{r = +q\}$; similarly, $\kappa(s,t)$ cannot satisfy KdV, $\kappa_t + \kappa\kappa_s + \kappa_{sss} = 0$, which would require $q = \kappa$, $r = -1$.

In addition to *local* equations like (10), (7) also admits *nonlocal* models, such as the sine-Gordon equation,

$$\theta_{st} = \sin\theta. \tag{11}$$

To obtain (11), choose $U = \Omega^{-2}\kappa_s$, so that (7) becomes $\Omega\kappa_t = \kappa_s$. This equation can be integrated once in s:

$$\kappa_{st} + \kappa\int_{-\infty}^s dz\,\kappa\kappa_t = \kappa. \tag{12}$$

Define $\theta(s,t) = \int_{-\infty}^s dz\,\kappa(z,t)$, and define $F(s,t)$ by $\partial\kappa/\partial t = \sin\theta + F$, so that $\theta_{st} = \sin\theta + F$. Substituting these into (12) yields $F_s + \theta_s\int_{-\infty}^s dz\,\theta_z F = 0$, which can be written as

$$\frac{dF}{d\theta} + \int_0^\theta d\theta'\, F = 0, \tag{13}$$

with the general solution $F = A\cos\theta + B\sin\theta$. For any choice of (A,B) one obtains $\theta_{st} = C\sin(\theta+\theta_0)$, which can be rescaled into (11).

Lamb [5] had obtained (11) as an equation for the three-dimensional motion of curves with constant nonzero torsion. The derivation given above shows that the curve can have zero torsion, and the motion can be purely two dimensional.

Because (9) requires (8) only at $\zeta = 0$, rather than for all ζ, it is not necessary to choose U so that (7) is integrable. Nor is it necessary to preserve the global invariants emphasized in Ref. [4], which we discuss next. If the curve has finite length initially (say, $0 \le \alpha \le 1$ at $t = 0$), then it follows from (6) that L, the total length of the curve, is a constant of the motion provided that (i) $W(\alpha = 1,t) = W(\alpha = 0,t)$, and (ii) $\int_0^L \kappa U\,ds = 0$. Independently, if $\int_0^L U\,ds = 0$, then $A := \int_0^L \boldsymbol{r}\times\partial\boldsymbol{r}/\partial s\,ds$ is conserved as well. If the curve is closed, then A represents its enclosed area. If $\kappa(s,t)$ is a periodic solution of (10), or of any of the other integrable motions in the mKdV hierarchy, then these two quantities are among infinitely many that are conserved in time. Among the nonintegrable motions (i.e., among all other possible 2D motions of curves), there exist motions that conserve L, or A, or neither, or both. Here are some examples. (i) $U = \partial\ln\kappa/\partial s$, $W = \kappa$. This motion is not in the mKdV hierarchy, but if $\kappa(s,t)$ is periodic in s and nonvanishing, then the motion conserves both L and A. If κ is not periodic in s, then the motion conserves L, but not A. (ii) $U = -\partial^2\kappa/\partial s^2$, $W = 0$. If κ is periodic in s, then the motion conserves A, but not L. If κ is not periodic in s, then neither L nor A is conserved. (iii) $U = \kappa$, $W = 0$. This leads to the so-called "curve-shortening equation" [9]. More generally, let $U = \sum_n c_n(-1)^n\partial^{2n}\kappa/\partial s^{2n}$, $W = 0$. It follows from (6) that for any choice of non-negative $\{c_n\}$, one obtains in this way a curve-shortening equation.

As a final comment about 2D motion, we mention that most solutions of (7) do *not* represent the motions of closed curves, even under the most favorable circumstances; namely, even if (i) U is chosen to give an integrable motion from the mKdV hierarchy, and (ii) W is chosen to preserve local arclength, so $\partial W/\partial s = \kappa U$, and (iii) $\kappa(s,t)$ is periodic in s, for all t.

To see this, let $\kappa(s,t)$ be a periodic solution of (7), with period L. The orientation of the curve relative to a fixed coordinate system is specified by $\theta(s,t)$, where $\kappa(s,t) = \partial\theta/\partial s$. A smooth, closed curve requires that

$$\theta(L,t) - \theta(0,t) = \int_0^L \kappa\, ds = 2\pi N. \qquad (14)$$

If (14) holds at $t = 0$, and if $\kappa(s,t)$ is periodic, then (14) continues to hold for $t \neq 0$, because (7) is itself a conservation law ($\partial\kappa/\partial t = \partial F/\partial s$).

In addition, a closed curve also needs

$$\boldsymbol{r}(s = L,t) - \boldsymbol{r}(s = 0,t) = \int_0^L \boldsymbol{t}\, ds = 0. \qquad (15)$$

In a fixed coordinates system, $\boldsymbol{t}$ has components $(\cos\theta, \sin\theta)$, so (15) implies that

$$\int_0^L e^{i\theta}\, ds = 0 \quad \text{or} \quad \oint \frac{e^{i\theta}}{\kappa}\, d\theta = 0. \qquad (16)$$

Even if κ is periodic in s and if θ satisfies (14), (16) imposes extra conditions that typically are not satisfied; i.e., most periodic solutions of (7) do not correspond to closed curves.

As a simple example, let us seek all closed curves whose shapes are invariant under mKdV, (10). Then $\kappa(s,t)$ must be a periodic solution of (10) in the form of a traveling wave, so $\kappa(s,t) = \kappa(x)$ where $x = s - \lambda t$, and

$$-\lambda\kappa' + (3/2)\kappa^2\kappa' + \kappa''' = 0. \qquad (17)$$

Obviously κ = const solves (17), and every circle is a closed invariant curve under (10). More generally, one integrates (17) twice to obtain

$$(\kappa')^2 + \tfrac{1}{4}(\kappa - c_1)(\kappa - c_2)(\kappa - c_3)(\kappa - c_4) = 0,$$
$$c_1 + c_2 + c_3 + c_4 = 0. \qquad (18)$$

We have found only one nontrivial class of closed invariant curves, corresponding to $c_1 = -c_4 = a$, $c_2 = -c_3 = ib$ (a,b are real):

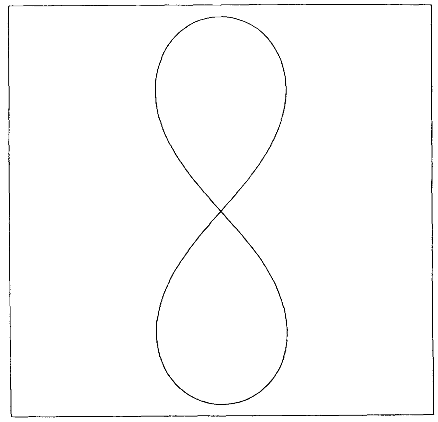

FIG. 1. Figure-8 shaped curve.

$$\kappa(x) = a\,\mathrm{cn}\,(\alpha(x - x_0), k), \quad k = \frac{a}{\sqrt{a^2 + b^2}},$$
$$\alpha = \tfrac{1}{2}\sqrt{a^2 + b^2}, \qquad (19)$$

where $\mathrm{cn}(x,k)$ is Jacobi's elliptic function [10]. From (16), the curve is closed only if the elliptic modulus k satisfies $E(k) = 2K(k)$, so that $k = 0.908\,911\ldots$. The corresponding curve is the figure-8 shape shown in Fig. 1. In the soliton limit, $k \to 1$ in (20), and the curve becomes a loop [11].

It is interesting to note that this family of curves which are shape invariant under (7) were first analyzed by Euler (1744) in his study of *elastica* [12]. The figure-8, the loop, and other possible shapes of elastica are illustrated in Ref. [12].

Three-dimensional motion.—The situation is similar for the motion of curves in 3-space, but there is more algebra. The generalization of (6) for 3D motions is

$$\begin{aligned}
\dot{\boldsymbol{t}} &= \left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\boldsymbol{n} + \left(\frac{\partial V}{\partial s} + \tau U\right)\boldsymbol{b},\\
\dot{\boldsymbol{n}} &= -\left(\frac{\partial V}{\partial s} - \tau V + \kappa W\right)\boldsymbol{t} + \left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right]\boldsymbol{b},\\
\dot{\boldsymbol{b}} &= -\left(\frac{\partial V}{\partial s} + \tau U\right)\boldsymbol{t} - \left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right]\boldsymbol{n},\\
\dot{g} &= 2g\left(\frac{\partial W}{\partial s} - \kappa U\right),
\end{aligned} \qquad (20)$$

where $(\cdot) = \partial(\ \ )/\partial t|_\alpha$, as before. Because $\dot{f}(s,t) = \partial f/\partial t + (W - \int^s \kappa U\, ds')\partial f/\partial s$, it follows that $\kappa(s,t)$ and $\tau(s,t)$ satisfy

$$\begin{aligned}
\frac{\partial\kappa}{\partial t} &= \frac{\partial^2 U}{\partial s^2} + (\kappa^2 - \tau^2)U + \frac{\partial\kappa}{\partial s}\int^s \kappa U\, ds' - 2\tau\frac{\partial V}{\partial s} - \frac{\partial\tau}{\partial s}V,\\
\frac{\partial\tau}{\partial t} &= \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V\right) + \tau\int^s \kappa U\, ds'\right] + \kappa\tau U + \kappa\frac{\partial V}{\partial s}\,.
\end{aligned} \qquad (21)$$

Because the curve can be reconstructed (up to orientation and translation) from a knowledge of $\kappa(s,t)$ and $\tau(s,t)$, (21) determines motion of the curve in 3-space. Notice again that $W(s,t)$ does not affect the evolution of the curve.

The relation of these 3D motions to integrable evolution equations is similar to that of 2D motions. We show next that the Serret-Frenet equations, (3), are equivalent to (8) at $\zeta = 0$, with $r = -q^*$. Hence, equations from the mKdV hierarchy with $\{r = -q^*\}$ represent possible motions of smooth curves in 3-space. The description in terms of NLS [1] is the simplest nontrivial case of an integrable motion. However, nonintegrable motions are possible as well.

According to (3), each set of components of $\{\boldsymbol{t}, \boldsymbol{n}\, \boldsymbol{b}\}$ satisfies

$$\frac{\partial t_j}{\partial s} = \kappa n_j, \quad \frac{\partial n_j}{\partial s} = -\kappa t_j + \tau b_j\,, \qquad \frac{\partial b_j}{\partial s} = -\tau n_j, \quad j = 1,2,3. \tag{22}$$

These equations admit an integral, which we normalize to $t_j^2 + n_j^2 + b_j^2 = 1$, $j = 1,2,3$. Following Lamb [5] and Darboux [13], we define

$$\phi(s,t) := \kappa(s,t)\varepsilon, \tag{23}$$

where $\varepsilon := \exp\{i \int^s \tau(s',t)\, ds'\}$. Similarly, for each $j = 1,2,3$, let

$$N_j := (n_j + i b_j)\varepsilon$$

so that (22) becomes

$$\frac{\partial N_j}{\partial s} = -\phi t_j, \quad \frac{\partial t_j}{\partial s} = \frac{1}{2}(\phi N_j^* + \phi^* N_j). \tag{24}$$

Then under the additional transformation,

$$w_1 = N_j \exp\left\{\frac{1}{2}\int^s \left(\frac{\phi N_j^*}{1 - t_j}\right) ds'\right\},$$

$$w_2 = (1 - t_j) \exp\left\{\frac{1}{2}\int^s \left(\frac{\phi N_j^*}{1 - t_j}\right) ds'\right\},$$

(24) becomes

$$\frac{\partial w_1}{\partial s} = \frac{1}{2}\phi w_2, \quad \frac{\partial w_2}{\partial s} = -\frac{1}{2}\phi^* w_1. \tag{25}$$

These are equivalent to (8) at $\zeta = 0$, with $q = -\phi/2$, $r = -q^*$, as asserted above. The realization that (3) implies (8) *only* at $\zeta = 0$ is the main difference between our results and those of Lamb [5].

Using (23), the evolution equations (21) become

$$\frac{\partial \phi}{\partial t} = \left[\frac{\partial^2}{\partial s^2} + |\phi|^2 + i\phi \int^s ds'\, \tau\phi^* + \frac{\partial \phi}{\partial s}\int^s ds'\, \phi^*\right](U\varepsilon) + \left[i\frac{\partial^2}{\partial s^2} + i|\phi|^2 + \phi\int^s ds'\, \tau\phi^* - i\phi\int^s ds'\, \frac{\partial \phi^*}{\partial s'}\right](V\varepsilon). \tag{26}$$

As an example of an integrable motion, let $U = 0$, $V = \kappa$. Then (26) reduces to the focusing version of NLS, $i\phi_t + \phi_{ss} + (1/2)|\phi|^2\phi = 0$. However, no choice of $\{U, V\}$ could give the defocusing version, $i\phi_t + \phi_{ss} - (1/2)|\phi|^2\phi = 0$, because this would require $\{r = +q^*\}$ in (8). As another example, if $U = -\kappa_s$, $V = -\kappa\tau$, then (26) reduces to the complex mKdV,

$$\phi_t + \phi_{sss} + (3/2)|\phi|^2\phi_s = 0. \tag{27}$$

If ϕ is initially real, corresponding to a 2D curve without torsion, then (27) reduces to (10), and the curve remains in the plane.

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