Improving Large-Order Perturbative Expansions in Quantum Chromodynamics

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We consider divergences of the perturbative expansions in large orders in quantum electro- and chromodynamics and concentrate on the dependence of large-order contributions on the choice of the normalization point of coupling constants. We find that for sign-alternating series the predictive power of perturbation theory as measured by the minimal term of an asymptotic expansion can be drastically improved by a proper choice of the normalization point. In particular, this allows the elimination of the leading uncertainty of perturbative expansions in quantum chromodynamics.

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Recently large-order estimates in quantum chromodynamic (QCD) perturbation theory have received new impetus [1–3]. One of the main issues discussed is to what extent the perturbative predictions are plagued by the asymptotical nature [4] of the perturbative expansions.

In particular, we shall be concerned with the imaginary part of the electromagnetic current correlation function

$$(-i)\int d^4x\, e^{iqx}\langle 0|T(j_\mu(x)j_\nu(0))|0\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)$$

in QCD and quantum electrodynamics (QED) with N massless fermion flavors. In QCD it is related to the total hadronic cross section in e^+e^- annihilation by

$$R(q^2)=\frac{\sigma_{e^+e^-\to \text{hadrons}}}{\sigma_{e^+e^-\to\mu^+\mu^-}} = 12\pi\,\mathrm{Im}\Pi\left(\frac{q^2}{\mu^2},\alpha_{\rm QCD}(\mu)\right)+O(\alpha_{\rm QED})$$

and is therefore of direct physical meaning.

The leading large-order contributions come from the so-called renormalons [5], which are potentially most important as limiting the predictive power of perturbative expansions. Namely, a generic perturbative expansion in the QCD running constant looks like

$$f(\alpha_{\rm QCD}(Q))=\sum p_n\alpha_{\rm QCD}^n(Q)\,, \tag{1}$$

where $f(\alpha_{\rm QCD})$ is, say, $\Pi(Q^2)$ ($Q^2=-q^2$ denote Euclidean momenta). The asymptotics of the expansion coefficients p_n is controlled at large n by the ultraviolet renormalon:

$$\lim_{n\to\infty} p_n = \text{const}\times b_0^n n! n^d\,,$$

where d is some constant and b_0 is the first coefficient in the β function so that b_0 is negative in QCD. Dealing with (1) as with ordinary asymptotic series, one concludes that the perturbative series approximates the physical quantity to accuracy no better than [3]

$$\delta\Pi(Q^2)=\text{const}\times\frac{\Lambda_{\rm QCD}^2}{Q^2}\,,$$

where $\Lambda_{\rm QCD}$ is the position of the pole of the running coupling (see below). Thus one comes to consider Q^{-2} terms which are usually omitted from the phenomenological analysis (see [6] and references therein). Note also that Q^{-2} terms were also introduced in [2] on general grounds.

So far we discussed expansions in $\alpha_{\rm QCD}$, as if the latter were uniquely defined. In fact the definition of $\alpha_{\rm QCD}$ is subject to an arbitrariness due to the choice of renormalization scheme (RS) and normalization point (scale). This scheme-scale ambiguity is a well-known problem in low orders [7]. In case of infrared stable quantities such as $\Pi(Q^2)$, scale dependence is usually dealt with by normalizing the coupling parameter at Q^2 which absorbs all logarithms into the definition of the coupling. Also, it has become conventional to perform calculations in the modified minimal subtraction ($\overline{\rm MS}$) scheme, which is known to give reasonable coefficients in low orders. As far as large orders are concerned, the use of $\alpha_{\rm QCD}(Q)$ is tacitly assumed to be unique.

In this Letter we will demonstrate that an alternative choice of normalization point can improve crucially the accuracy of the perturbative expansions and allows, in particular, the effective suppression of the Q^{-2} uncertainty in the QCD case mentioned above.

Although our main concern is QCD, we shall concentrate here mostly on the simpler but similar case of QED. In QED it is convenient to rescale the coupling, $a(\mu)=\alpha(\mu)N$, and represent $\Pi(Q^2)$ as a sum:

$$\Pi(Q^2,\mu^2,a(\mu),N)=\sum_{p=-1}^{\infty}\Pi_p(Q^2,\mu^2,a(\mu))\frac{1}{N^p}\,.$$

Each fermion loop brings a factor of N and thus Π_p is contributed by all graphs with $\frac{1}{2}\times$number of vertices $-$ number of fermion loops $=p$. We will consider the perturbative expansion of Π_0,

$$\Pi_0(Q^2,\mu^2,a(\mu))=\sum_{n=0}^{\infty}\Pi_0^{(n)}(Q^2,\mu^2)a(\mu)^{n+1}\,,$$

which is given by the graphs with a string of n vacuum bubbles inserted as in Fig. 1 and the counterterms to make these graphs finite. It is precisely these graphs which yield the leading factorial-like divergence of the

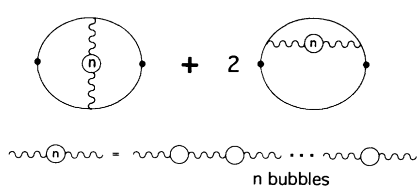

FIG. 1. Renormalon-type diagrams that contribute to $\Pi_0^{(n)}$.

series in large orders [5,8].

The simplicity of QED is that, owing to the Ward identity, to leading order a scheme and scale invariant effective charge can be defined in QED in terms of a single constant C [9]:

$$\frac{1}{a_{\rm eff}(Q)}=\frac{1}{a(\mu)}-b_0\left(\ln\frac{Q^2}{\mu^2}+C\right)+O\left(\frac{a}{N}\right). \tag{2}$$

Moreover, the constant C is introduced through the renormalization of the one-loop vacuum polarization,

$$\pi(Q^2,\mu^2,a(\mu))=-b_0a(\mu)\left(\ln\frac{Q^2}{\mu^2}+C\right)+O\left(\frac{a^2}{N}\right). \tag{3}$$

In the $\overline{\rm MS}$ scheme, e.g., $C=-\frac{5}{3}$. Defining the scale invariant Λ in a particular scheme as the pole of the coupling

$$a(\mu)=\frac{1}{b_0\ln(\Lambda^2/\mu^2)},\quad \mu^2\ll\Lambda^2,$$

we conclude from (2) that Λ^2e^{-C} is also an RS invariant [10].

The Borel transform of $\mathrm{Im}\Pi_0$ can be calculated explicitly with the result

$$\mathrm{Im}\Pi_0\left(\frac{q^2}{\mu^2},a(\mu)\right)=\int_0^\infty dt\, e^{-t/a(\mu)}B[\mathrm{Im}\Pi_0]\left(\frac{q^2}{\mu^2},t\right)=\int_0^\infty dt\exp\left\{-t\left[\frac{1}{a(\mu)}-b_0\left(\ln\frac{q^2}{\mu^2}+C\right)\right]\right\}D(b_0t)\sin(\pi b_0t), \tag{4}$$

where $D(u)$ is a quite complicated function whose precise form is not important for the present purpose [11]. The only piece of information we shall need is that $D(u)$ indeed exhibits the expected renormalon singularities at integer u.

The exponent in (4) is naturally expressed in terms of the effective charge (2). The Borel sum is thus RS and normalization point invariant, but we emphasize that this is a purely formal statement, since the integral does not exist due to the poles of D on the positive axis [12]. But note that even if the Borel sum of Π_0 is ill defined, the representation (4) may still be viewed as a concise form of the original series in $a(\mu)$, which is recovered by expanding the Borel transform in t and integrating term by term. This yields

$$\Pi_0^{(n)}\left(\frac{Q^2}{\mu^2}\right)=\frac{d^n}{dt^n}B[\Pi_0]\left(\frac{Q^2}{\mu^2},t\right)\Bigg|_{t=0}.$$

To calculate the derivatives for large n, we appeal to Darboux' theorem (see, e.g., [13]), which states that high-order derivatives are determined by the singularities of the function under consideration closest to the origin. We discuss here two cases separately: (i) Infrared (IR) renormalon so that $D(u)$ has a simple pole at $u=-2$. (ii) Ultraviolet (UV) renormalon so that $D(u)$ has a simple pole at $u=+1$ [14].

(i) We thus suppose that near $u=-2$, $D(b_0t)\sim A/(2+b_0t)$, A being some constant. Inserting this into (4) (without the sin factor) gives the series

$$\Pi_0\left(\frac{Q^2}{\mu^2},a(\mu)\right)=\sum_{n=0}^{\infty}(-1)^n\frac{A}{2}\left(\frac{b_0}{2}\right)^n\exp_n\left[-2\left(\ln\frac{Q^2}{\mu^2}+C\right)\right]n!a(\mu)^{n+1}$$

to be trusted for large n and $\exp_n(x)\equiv\sum_{k=0}^{n}x^k/k!$. Since we are interested in large n, we may replace $\exp_n(x)$ by $\exp(x)$. Moreover, in the case of sign-alternating series the best accuracy of approximation of the true result through an asymptotic expansion is estimated by the term least in absolute magnitude [13]. Thus we get

$$\delta\Pi_0\left(\frac{Q^2}{\mu^2},a(\mu)\right)\sim A\left(\frac{\pi a(\mu)}{|b_0|}\right)^{1/2}\exp\left[-2\left(\ln\frac{Q^2}{\mu^2}+C\right)\right]\exp\left(-\frac{2}{|b_0a(\mu)|}\right)=A\left(\frac{\pi a(\mu)}{|b_0|}\right)^{1/2}\left(\frac{Q^2}{\Lambda^2e^{-C}}\right)^2\left(\frac{\mu^2}{Q^2}e^{-C}\right)^4 \tag{5}$$

with $n_{\rm min}\sim 2/|b_0|a(\mu)$.

Recalling that Λ^2e^{-C} is RS invariant, $\delta\Pi_0$ can be made arbitrarily small for any Q^2 by choosing either small μ or a scheme with large C. Thus upon the change of the expansion parameter in sign-alternating asymptotic series one can improve the accuracy of the perturbative expansion. It might also be worth emphasizing that our results are valid without assuming Borel summability but performing all transformations with a finite number of terms in the series.

The lesson to be learned is that $a(Q)$ need not be the best choice of expansion parameter from the perspective of large-order perturbation theory. In particular, by proper choice of scheme and normalization point, the ultimate divergence of perturbation series will be entirely due to the first singularity on the positive axis; see (ii) below.

(ii) Now, near $b_0t=1$, $D(b_0t)\sim A'/(1-b_0t)$, producing a single-sign series. The minimal term of the perturbative expansion of Π_0, derived from this as above, is estimated as

$$\delta\Pi_0\left[\frac{Q^2}{\mu^2},a(\mu)\right]\sim A'\left[\frac{2\pi a(\mu)}{b_0}\right]^{1/2}\frac{Q^2}{\Lambda^2 e^{-C}}.$$

We see that in this case $\delta\Pi_0$ is essentially RS and scale independent apart from the slowly varying factor $\sqrt{a(\mu)}$. Moreover, now there is no general relation between the minimal coefficient and the best accuracy of approximation of the true value as for sign-alternating series. Thus it is much more appropriate to use the ambiguity in the definition of the Borel integral, given by the residue of the pole,

$$\mathrm{res}_{b_0t=1}=\frac{A'}{b_0}\frac{Q^2}{\Lambda^2 e^{-C}},$$

as limiting the precision of perturbation theory. This term is scheme and scale invariant.

Thus, the ambiguities of the perturbative series associated with poles of the Borel transform on the positive axis cannot be decreased by choice of scheme or normalization point.

We now turn to the phenomenologically much more interesting case of QCD. Note however, that, at least as the b_0 dependence of renormalons is concerned, QCD is very much like QED. This is illustrated by the fact that in QCD the renormalon singularities of $B[\Pi]$ still fall at n/b_0, $n=1,\pm 2,\ldots$ [5], although the class of diagrams which contributes to this result is not exhausted by the graphs of Fig. 1 and b_0 is no longer the coefficient of the one-loop gluon vacuum polarization as in (3). Also, the exponent in the Borel representation (4) is always organized such as to absorb the Q^2 dependence into the inverse running coupling. This can easily be verified by exploiting the renormalization-group equation for the Borel transform. Therefore, the conclusions drawn from the previous section carry over to QCD, at least as far as variation of the normalization point is concerned (C, of course, no longer has the simple meaning as in QED). The only thing we should do is to substitute for b_0 its value in QCD. But since now b_0 is negative, the roles of infrared and ultraviolet renormalon singularities become interchanged. The latter are Borel summable, whereas the former are not.

This has interesting consequences for the following reason: in QCD the first UV renormalon occurs at $-1/|b_0|$, which is closer to the origin of the Borel plane than the first IR renormalon at $2/|b_0|$. Thus, the onset of divergence of the perturbative series in terms of $\alpha_s(Q)$ is due to the UV renormalon and shows sign alternation. If, however, we follow the above considerations, then by choosing a proper normalization point the divergence of the series can be delayed to IR singularities. Note that in the QCD case we would have to take large μ, $\mu\gg Q$, since now the analog to (5) reads

$$\delta\Pi_{\mathrm{UV}}\left[\frac{Q^2}{\mu^2},a(\mu)\right]\sim A\left[\frac{2\pi a(\mu)}{|b_0|}\right]^{1/2}\frac{\Lambda^2}{Q^2}\left[\frac{Q^2}{\mu^2}\right]^2$$

(scheme dependence has been absorbed into A). In particular, the minimal term of the series can be decreased in this way and the accuracy of perturbative predictions improved, until one runs into the first IR singularity, which produces a Q^{-4} ambiguity.

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[11] The definition of the Borel transform of a power series $f=\sum_{n=0}^{\infty}f_n a^{n+1}$ we use is $B[f](t)\equiv\sum_{n=0}^{\infty}(f_n/n!)t^n$. The details of the derivation of (4) as well as discussion of the singularities of D will be the subject of a separate publication [M. Beneke (to be published)]. The sin factor is due to analytical continuation of Π_0 from Euclidean to Minkowskian momenta as has been emphasized in a general context in [2]. Thus Π_0 itself has a similar Borel representation with the sin omitted and up to an unimportant subtraction constant (which could be eliminated by considering $-q^2 d/dq^2\Pi_0$). In the following we consider Π_0, ignoring this subtraction.

[12] For a discussion of the QCD case, see [5].

[13] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic, London, 1973).

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