

Renormalization-Group Trajectories from Resonance Factorized S Matrices

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We propose and investigate a large class of models possessing resonance factorized S matrices. The associated Casimir energy describes a rich pattern of renormalization-group trajectories related to flows in the coset models based on the simply laced Lie algebras. From a simple resonance S matrix satisfying the “ ϕ^3 property” we predict new flows in nonunitary minimal models.

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Recently Al Zamolodchikov introduced the “staircase model” [1], which is defined by an S matrix of a single massive particle m with amplitude $S(\theta, \theta_0) = [\sinh(\theta) - i \cosh(\theta_0)] / [\sinh(\theta) + i \cosh(\theta_0)]$. The important characteristic of this S matrix is that it possesses two resonance poles at $\theta = -i\pi/2 \pm \theta_0$. The ultraviolet behavior is governed by a theory of central charge $c = 1$. At intermediate distances, however, this model exhibits a rather rich pattern of renormalization-group (RG) trajectories for variation of the real parameter θ_0 . For θ_0 large enough, Zamolodchikov [1] has found that the associated Casimir energy (defined in a torus of radius R) interpolates between the central charges $c_p = 1 - 6/p(p+1)$, $p = 3, 4, \dots$, of the minimal models M_p . Roughly, the Casimir energy $E_0(R, \theta_0)$ forms plateaus of approximate length $\theta_0/2$ in each value of c_p before smoothly crossing over to the next lower fixed point c_{p-1} . In the literature the flow $M_p \rightarrow M_{p-1}$ is also known [2,3] as the effect of the $\phi_{1,3}$ perturbation to the critical point M_p .

It is important to stress that the universal properties of the critical point are highly dependent of the RG scenario in the vicinity of the criticality. Therefore, it seems relevant to extend the staircase model in order to include a more general class of flows. Also, pursuing this approach we may as well end up discovering new crossover behavior. This is precisely the purpose of this Letter. We start by proposing a resonance $Z(N)$ -factorizable scattering theory in which the behavior of its associated

RG trajectories will be related to a certain deformation of the minimal $W(A_{N-1})$ conformal models. It turns out that this theory is related to the scattering of the A_{N-1} Toda model [4] for complex values of its coupling constant. This relation allows us to easily conjecture the resonance scattering of the D and E Lie algebras. This last relation is only formal, due to the fact that the Toda Lagrangian needs an extra meaning for complex values of its coupling constant. However, the associated scattering theory is perfectly well defined, producing typical RG trajectories of flows in the deformed coset models based on simply laced Lie groups. As a by-product of our generalization we predict new flows in nonunitary minimal models from the simplest resonance scattering theories satisfying the “ ϕ^3 property”.

We first start by describing the resonance $Z(N)$ scattering. The spectrum consists of a set of particles and antiparticles with masses $m_i = \sin(i\pi/N)/\sin(\pi/N)$, $i = 1, 2, \dots, N-1$ [5]. The antiparticle appears in the particle-particle amplitude, and the factorizability implies [6] that the only constraints are the crossing and the unitarity conditions,

$$S_{i,j}(\theta)S_{i,j}(-\theta) = 1, \quad S_{i,j}(\theta) = S_{N-j,i}(i\pi - \theta), \quad (1)$$

where $N-j$ is the antiparticle of the j particle.

There is a family of solutions of Eq. (1) as a function of a real parameter θ_0 , which is responsible for the resonance poles. The minimal solution reads

$$S_{1,1}(\theta, \theta_0) = \frac{\sinh \frac{1}{2}(\theta + i2\pi/N) \sinh \frac{1}{2}(\theta - \theta_0 - i\pi/N) \sinh \frac{1}{2}(\theta + \theta_0 - i\pi/N)}{\sinh \frac{1}{2}(\theta - i2\pi/N) \sinh \frac{1}{2}(\theta - \theta_0 + i\pi/N) \sinh \frac{1}{2}(\theta + \theta_0 + i\pi/N)}. \quad (2)$$

The physical pole is at $\theta = i2\pi/N$, while the resonance poles appear in the unphysical sheet at $\theta = -i\pi/N \pm \theta_0$. The other amplitudes S_{ij} are obtained from $S_{1,1}$ by applying the bootstrap approach at $\theta = (i\pi/N)[i - j + 2(a - b)]$, $a = 1, 2, \dots, j-1$, $b = 1, 2, \dots, i-1$. For $N=2$, we obtain Zamolodchikov’s model [1].

Here our interest is to study this theory at intermediate distances, by analyzing the finite-volume effects to the Casimir energy. An effective way to study the Casimir energy $E(R, \theta_0)$ in a geometry of finite volume R is via the thermodynamic Bethe ansatz (TBA) approach [7,8]

at temperature $T = 1/R$. For the sake of simplicity let us first concentrate on the case $N=3$. In this case the Casimir energy $E(R, \theta_0)$ is given by

$$E(R, \theta_0) = -\frac{m}{\pi} \int_{-\infty}^{\infty} d\theta \cosh(\theta) L(\epsilon), \quad (3)$$

where $L(\epsilon) = \ln(1 + e^{-\epsilon(\theta)})$, m is the mass of the particle and its antiparticle, and the pseudoenergy $\epsilon(\theta)$ satisfies the following integral equation:

$$\epsilon(\theta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \psi(\theta - \theta', \theta_0) L(\epsilon) = mR \cosh(\theta), \quad (4)$$

where $\psi(\theta, \theta_0) = -i(d/d\theta) \ln[S_{1,1}(\theta, \theta_0)S_{1,1}(i\pi - \theta, \theta_0)]$.

The ultraviolet limit of the Casimir energy, $R \rightarrow 0$, is independent of θ_0 and we find the behavior $E(R, \theta_0) \simeq -2\pi/6R$, which implies that the background conformal theory has central charge $c=2$ [9]. In order to analyze the behavior of the function $c(R, \theta_0) = -6R \times E(R, \theta_0)/\pi$ for finite values of R we numerically solved Eq. (2), in the convenient variable $X = \ln(mR/2)$, by standard interactive procedure. For $\theta_0=0$, $c(R, 0)$ behaves as a smooth function between the ultraviolet ($c=2$) and the infrared ($c=0$) regimes. By increasing θ_0 , however, we observe that certain plateaus start to form precisely around the values that parametrize the central charge of the minimal model of the $W(A_2)$ algebra, namely, $c=2[1 - 12/p(p+1)]$, $p=4, 5, \dots$. For example at $\theta_0=40$, we notice at least 8 plateaus starting at $p=12$ and subsequently visiting the other fixed points $p=11, \dots, 4$, until finally reaching the infrared region. In Figs. 1(a) and 1(b) we show this behavior for $\theta_0=20, 40$. The same pattern can be viewed from the beta function along the RG trajectories. Following Zamolodchikov's notation [1], one can define the beta function as

$$\beta(g) = -\frac{d}{dX} c(R, \theta_0), \quad g = 2 - c(R, \theta_0). \quad (5)$$

In Figs. 2(a) and 2(b), we show $\beta(g)$ for $\theta_0=20, 40$. The zeros of $\beta(g)$ are formed precisely at $g=24/p(p+1)$, $p=4, 5, \dots$, in accordance with the plateaus mentioned above.

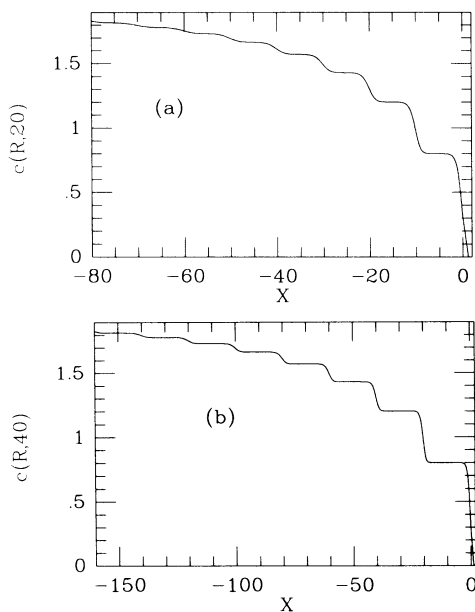


FIG. 1. The scaling function $c(R, \theta_0)$ for (a) $\theta_0=20$ and (b) $\theta_0=40$.

In the case of general N we should expect a similar behavior. From Zamolodchikov's discussion of $N=2$ and our present results we conclude that each time that $X \simeq -(p-N)\theta_0/2$ the function $c(R, \theta_0)$ will cross over its value of $c_p = (N-1)[1 - N(N+1)/p(p+1)]$, $p=N+1, N+2, \dots$, to the next fixed point with central charge c_{p+1} . Indeed, by linearizing the TBA equations around $X \simeq -(p-N)\theta_0/2$ one remains with the same equation that describes the flow in the $W(A_{N-1})$ minimal models perturbed by the least $Z(N)$ -invariant operator [10]. However, we stress that the bulk of each plateau has the approximate length of θ_0/N , in agreement with the fact that the finite-size corrections are N dependent.

As an important remark we mention that our proposed resonance $Z(N)$ -factorized model is easily connected to the one of the A_{N-1} Toda field theory [4], by making an analytical continuation to the complex values of the Toda coupling constant. The coupling constant α enters in the S matrices through a function $b_{\text{Toda}}(\alpha)$ [4]. By setting $b_{\text{Toda}}(\alpha) = \pi/N \pm i\theta_0$ in Eq. (2), we recover the minimal S matrix of the A_{N-1} Toda model [4]. This leads us to conjecture that the resonance scattering theories, based on the simple laced Lie algebras A, D, E can be obtained from the corresponding Toda S matrices [11]. The resonance parameter θ_0 is introduced through the simple rela-

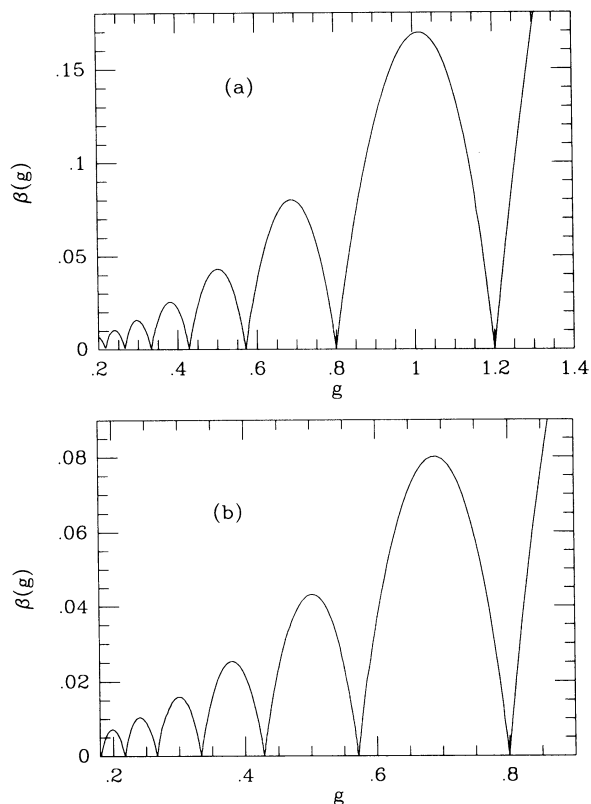


FIG. 2. The beta function $\beta(g)$ for (a) $\theta_0=20$ and (b) $\theta_0=40$. For $\theta_0=40$ we have omitted the first zero at $p=5$ in order to better show the remaining zeros of $\beta(g)$.

tion $b_{\text{Toda}}(\alpha) = \pi/h \pm i\theta_0$, where h is the dual Coxeter number of the respective A, D, E Lie algebra. Using this last relation it is possible to express the two-body resonance S matrices $S_{a,b}(\theta, \theta_0)$ by the equation

$$S_{a,b}(\theta, \theta_0) = \exp \left(\frac{i}{\pi} \int_0^\infty \psi_{a,b}(k, \theta_0) \frac{\sin(\theta k)}{k} dk \right), \quad (6)$$

where $\psi_{a,b}(k, \theta_0)$ satisfy the remarkable matrix identity

$$\left[\delta_{a,b} - \frac{\psi_{a,b}(k, \theta_0)}{2\pi} \right]^{-1} = \frac{\delta_{a,b} \cosh(\pi k/h) - l_{a,b}/2}{\cosh(\pi k/h) - \cos(k\theta_0)} \quad (7)$$

and $l_{a,b}$ is the incident matrix of the respective A, D, E Lie algebra.

We have also analyzed the TBA equations obtained from Eqs. (6) and (7) around $X \simeq -(p-h)\theta_0/2$ and performed numerical checks. Our conclusion is that the A, D, E resonance scattering proposed in Eqs. (6) and (7) will produce RG trajectories associated with the flows in the coset model $G_{p-h} \otimes G_1 / G_{p-h+1}$ ($G = A, D, E$) [12] perturbed by the field Φ with conformal dimension $\Delta_\Phi = 1 - h/(p+1)$ [10]. The scaling corrections in the infrared regime are made by the ‘‘dual’’ operator $\tilde{\Phi}$ with conformal dimension $\Delta_{\tilde{\Phi}} = 1 + h/(p-1)$ (for $p = h+1$,

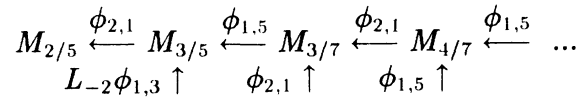


FIG. 3. The flow pattern in the nonunitary minimal models $M_{q/(2q+1)}$, $M_{(q+1)/(2q+1)}$, $q = 2, 3, \dots$. The horizontal (vertical) arrows represent the relevant (irrelevant) operators defining the ultraviolet (infrared) corrections to the fixed point.

this field is replaced by the spinless combination $T\bar{T}$ of the stress energy tensor T). The field Φ ($\tilde{\Phi}$) is the analog of the operators $\phi_{1,3}$ ($\phi_{3,1}$) of the minimal models. It has been argued [13] that the combination $\lambda\phi_{1,3} + \tilde{\lambda}\phi_{3,1}$ plays a fundamental role in the description of Zamolodchikov’s staircase model as a perturbed conformal field theory. In our case, the straightforward generalization will consider the combination $\lambda\Phi + \tilde{\lambda}\tilde{\Phi}$. We remark, however, that this picture has to be checked by a careful analysis of the finite-size corrections to the fixed point [14].

Let us now introduce a resonance scattering model possessing the ϕ^3 property that will be connected with new flows in nonunitary minimal models with $c < 1$. The model consists of a single particle a , and its two-body S matrix is given by

$$S_{a,a}(\theta, \theta_0) = \frac{\tanh \frac{1}{2}(\theta + i2\pi/3) \tanh \frac{1}{2}(\theta - \theta_0 - i\pi/3) \tanh \frac{1}{2}(\theta + \theta_0 - i\pi/3)}{\tanh \frac{1}{2}(\theta - i2\pi/3) \tanh \frac{1}{2}(\theta - \theta_0 + i\pi/3) \tanh \frac{1}{2}(\theta + \theta_0 + i\pi/3)}. \quad (8)$$

The pole at $\theta = i2\pi/3$ produces the particle itself (ϕ^3 property) and the resonance poles are located at $\theta_0 = i\pi/3 \pm \theta_0$. It turns out that the amplitude $S_{a,a}(\theta, \theta_0)$ satisfies the relation $S_{a,a}(\theta, \theta_0) = S_{1,1}(\theta, \theta_0)S_{1,2}(\theta, \theta_0)$, where $S_{i,j}(\theta, \theta_0)$ are the S matrices of the resonance $Z(3)$ model. The equivalent Toda theory [4] is the one proposed by Mikhailov [15] as a particular reduction of the $Z(3)$ Toda model. From the TBA point of view, this implies that the Casimir energy associated to $S_{a,a}(\theta, \theta_0)$ is precisely half of that of the $Z(3)$ model. Hence, now the plateaus will be formed around the values $c_p = 1 - 12/p(p+1)$, $p = 4, 5, \dots$. This result suggests that we are dealing with RG trajectories of the nonunitary minimal models. We recall that in this case the Casimir energy is identified with the effective central charge $c_{\text{eff}} = (-24\Delta + c)$, where Δ is the lowest conformal dimension [16]. Indeed, this is satisfied by the following classes of nonunitary minimal models:

$$M_{q/(2q+1)}, M_{(q+1)/(2q+1)}, q = p-2 = 2, 3, \dots \quad (9)$$

The flow pattern is shown in Fig. 3. We see that while in the $M_{q/(2q+1)}$ model the relevant (irrelevant) operator associated with the perturbation (infrared corrections) is $\phi_{1,5}$ ($\phi_{2,1}$), in the $M_{(q+1)/(2q+1)}$ theory the situation is replaced by $\phi_{2,1}$ ($\phi_{1,5}$). In the RG trajectories the fields $\phi_{1,5}$ and $\phi_{2,1}$ interchange their roles of relevant and irrelevant operators, building an extremely interesting pattern of

flow. For an extra support we also have numerically studied the spectrum of the simplest case, namely $M_{3/5} + \phi_{2,1} \rightarrow M_{2/5}$. Our results are compatible with the typical behavior expected of the RG flows. A consequence of this new crossover behavior is to impose constraints in the physical content of the related nonunitary minimal models. For example, it is possible to show [14] that the flow $M_{3/5} + \phi_{2,1} \rightarrow M_{2/5}$ is compatible with a Φ^6 Ginzburg-Landau action to the $M_{3/5}$ theory. Also, in the case of $M_{3/5}$, the field $\phi_{1,5}$ (not present on its Kac table) is substituted by the level-2 descendent of the $\phi_{1,3}$ operator.

In summary we have discussed rich classes of renormalization-group trajectories obtained from resonance scattering models based on the A, D, E Lie algebras. New flows in nonunitary minimal models have also been predicted. In our discussions the main bulk of technical details were omitted, and they will be presented in a forthcoming publication [14].

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