Nucleation and Transients at the Onset of Vortex Turbulence

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We present analytical and numerical results that explain the transient turbulent dynamics observed in the complex Ginzburg-Landau equation. Just below the transition to turbulence, we observe that metastable turbulent states break down by the nucleation and growth of single-vortex droplets, leading to a "frozen" state with a low (but finite) density of stationary vortices. We derive the relation between nucleation time and radius, and determine their dependence on the distance to the turbulence transition line.

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The appearance of turbulence (or "spatiotemporal chaos") in extended dynamical systems is not well understood. On the one hand, one would like to understand in what respects these systems behave like their low-dimensional, chaotic counterparts; on the other hand, it is obvious that, in some respects, they behave like stochastic systems, e.g., systems in thermal equilibrium. The onset of turbulence in media possessing a stable, time-periodic, global state has proven interesting since there one finds that the turbulence is associated with the appearance of "defects" (vortices or spiral waves) and thus one knows the important "elementary excitations." Vortex turbulence arises in oscillatory chemical media, and is characterized by a spatiotemporal chaotic dynamics, governed by the creation and annihilation of moving vortices [1-5]. A prototype for modeling this dynamics is the complex Ginzburg-Landau equation [6], which describes an oscillatory chemical medium in the vicinity of a Hopf bifurcation,

$$\dot{A}(\mathbf{x},t)=\mu A(\mathbf{x},t)-(1+i\alpha)|A(\mathbf{x},t)|^2A(\mathbf{x},t)+(1+i\beta)\nabla^2A(\mathbf{x},t)\,. \tag{1}$$

Here, $A(\mathbf{x},t)$, the complex amplitude at point $\mathbf{x}$ and time t, is related to the chemical mode $c(\mathbf{x},t)$ that develops the instability,

$$c(\mathbf{x},t)=c_0+[A(\mathbf{x},t)e^{i\omega_0 t}+\text{c.c.}]\,, \tag{2}$$

where c_0 is a constant, ω_0 is the oscillation frequency, and c.c. denotes the complex conjugate.

The parameters μ, α, and β in (1) are real numbers. For $\mu\le 0$, $A(\mathbf{x},t)=0$; for $\mu>0$, $A(\mathbf{x},t)$ is nonzero. In the latter case, a simple, homogeneous solution exists, $A(t)=\sqrt{\mu}e^{-i\alpha\mu t}$. Linear stability analysis shows that the homogeneous solution is stable when $\alpha\beta+1>0$. However, in general the system does not end up in the homogeneous state; rather it finds a "vortex state," i.e., one containing vortices and antivortices (spiral waves) with the property that the angle field changes by $\pm 2\pi n$ along any contour containing them. This state can be a "frozen" state of stationary vortices [2,3], or it can be a highly turbulent state with vortex-antivortex pair creation and annihilation.

Of fundamental interest is the occurrence of transient turbulent states near the transition line (in α-β space) that separates the frozen states from the turbulent states [2]. In this Letter, we present new analytical and numerical results on these transient states, which we identify as "metastable" states having a well-defined vortex density. We find that the metastable states break down through the nucleation and growth of single vortices, ending up in frozen states, which, due to the large system sizes in our simulations, we can identify as states with a low but finite density of stationary vortices. The lifetime of the metastable states depends on the distance to the transition line. We present arguments giving the analytical form of this dependence, and test it numerically.

As first noted by Oono and Puri [7], an efficient way to study Ginzburg-Landau models numerically is to construct a coupled-map lattice with a similar spatiotemporal dynamics. Such a construction has also been made for the complex Ginzburg-Landau model [2,5]. The idea is to integrate (1) in two steps by (i) solving the complex heat equation (taking only the Laplacian term into account), and (ii) integrating the nonlinear part neglecting the Laplacian. Here, we use the coupled-map lattice of Bohr *et al.* [2] adapted to the parallel architecture of the Boston University Connection Machine 2.

Figure 1 shows the phase diagram found for the complex Ginzburg-Landau model with $0\le\alpha\le 2$ and $-2\le\beta\le 0$, and solved on a square lattice of size 512×512. In the discrete model used, the (Benjamin-Feir) stability line for the homogeneous solution is

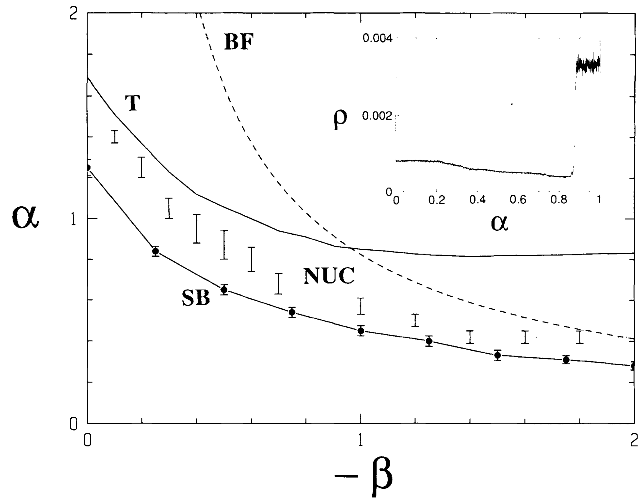

FIG. 1. Phase diagram obtained for the complex Ginzburg-Landau model with $\mu=1$, $0\leq\alpha\leq2$, and $-2\leq\beta\leq0$, and solved on a square lattice of size 512×512. —— (T): Transition line $\alpha_0(\beta)$ to vortex turbulence. --- (BF): Benjamin-Feir line. ···· (NUC): Line $\alpha_c(\beta)$ below which nucleation is no longer observed. -●- (SB): Line $\alpha_c(\beta)$ estimated by linear stability analysis. Inset: Vortex density $\rho(\alpha)$ for fixed $\mu=1$ and $\beta=-1$. ——: Density is obtained by slowly increasing α with a rate $d\alpha/dt=5\times10^{-6}$, starting from an (asymptotic) frozen state at $\alpha=0$. ---: Density for the metastable turbulent states.

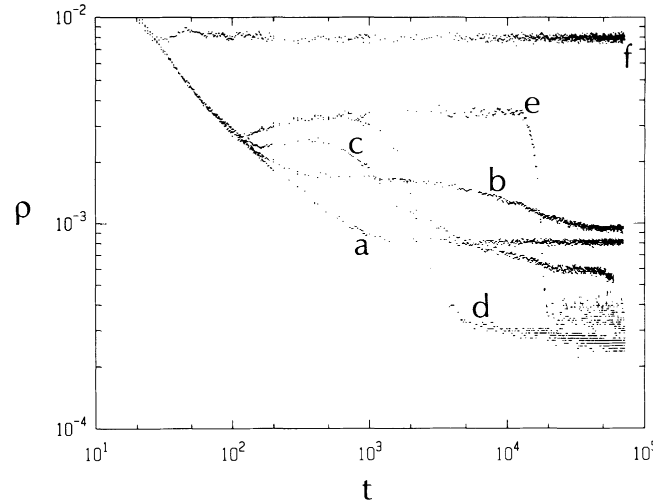

FIG. 2. Vortex density $\rho(t)$ obtained starting from a random initial state for fixed $\mu=1$ and $\beta=-1$. The α values are as follows: a, 0.0; b, 0.25; c, 0.6; d, 0.75; e, 0.8; and f, 2.0.

$\alpha\beta+\gamma=0$ (dashed curve "BF"), where [2] γ depends on both μ and the time step τ like $\gamma=(1-e^{-2\mu\tau})/2\mu\tau$, which approaches 1 as $\tau\rightarrow0$. Our simulations, including the ones used to generate Fig. 1, were done with $\mu=1$ and $\tau=0.2$ corresponding to $\gamma\simeq0.8242$. Starting from random initial conditions, the system does not end up in the homogeneous state, but in a vortex state. The solid curve ("T") in Fig. 1 is the transition line to turbulence, $\alpha=\alpha_0(\beta)$, which crosses the Benjamin-Feir line at $\beta\simeq-0.96$. Below the transition line, one finds asymptotically a low-density state of frozen, randomly distributed vortices separated by well-defined domain walls. Approaching the transition line, the (asymptotic) vortex density ρ slowly decays (inset to Fig. 1) through the breakdown and rearrangement of domain walls, followed by vortex-antivortex annihilations—the system again ending up in a frozen state. In the parameter range studied we find an almost constant minimal value of ρ, $\rho_{\min}\simeq(2.9\pm0.6)\times10^{-4}$. Spiral waves (vortices) that are formed in different regions of the medium all have the same parameter-dependent wavelength λ and oscillation frequency ω. Interestingly, we find that λ is close to constant, $\lambda=\lambda_0\simeq12$, along the transition line.

At the onset of turbulence, the density ρ increases abruptly by an order of magnitude, the frozen state breaks down, and a state with many randomly moving vortices is reached. This abrupt jump in density leads us to the question of what happens when the system is quenched from a turbulent state into the frozen regime. We have, therefore, followed the time evolution, starting from random initial conditions with many moving vortices. Figure 2 shows the vortex density ρ as a function of time t obtained for various values of α when $\beta=-1.0$.

Within a relatively short time $T_s\simeq100$, the vortex density ρ decays rapidly with negligible dependence on α. This initial decay ($t<T_s$) can be understood on the basis of random vortex motion, with a vortex-antivortex annihilation rate proportional to the density squared [4],

$$\frac{d\rho}{dt}\sim-\rho^2. \tag{3}$$

The solution of (3) is

$$\rho(t)\sim t^{-1}, \tag{4}$$

in agreement with our numerical results at early times.

We concentrate on the behavior at times $t>T_s$ for $\beta=-1$. Here, we find a qualitatively different vortex dynamics below $\alpha_c\simeq0.57$ than above α_c. For $\alpha<\alpha_c$, the vortex density slowly decays towards a constant value. The corresponding vortex state is basically frozen, but from time to time a domain wall breaks down resulting in vortex-antivortex annihilation.

For $\alpha>\alpha_c$, a density minimum occurs at T_s, after which the density stabilizes (at a value given by the dashed line in the inset of Fig. 1). The vortex density stays roughly constant until the time T_s+T, after which it decays to a stable value that characterizes the frozen state. We find that the transient state is turbulent with vortex-antivortex pair annihilations balancing pair creations. The average transient lifetime T of this "metastable" turbulent state diverges as we approach the transition line [8]. Just after time T_s+T, vortex "droplets" nucleate, as can be seen in Fig. 3, and the state eventually becomes a frozen, low-density pattern of vortices. Therefore, we think of the time T as a *nucleation time*—the time required to nucleate a growing droplet. The line $\alpha=\alpha_c(\beta)$, where the metastable turbulent dynamics first appears, is shown in Fig. 1 (dotted line "NUC"): Below here nucleation no longer occurs, because the domain-wall dynamics and the turbulent dynamics coincide at $t\sim T_s$.

The value $\alpha=\alpha_c(\beta)$ can be estimated by linear stability analysis (solid-circle line "SB" in Fig. 1) [2]: For $\alpha>\alpha_c$, the plane-wave solution to (1) with the selected, parameter-dependent wavelength is linearly unstable (sideband instability) [9]. As the figure shows, this estimate is always lower than the one found from nucleation. This is to be expected since the magnitudes of the eigenvalues determining the instability grow very slowly with α above the instability line, and the corresponding nucleation times T become too short to be resolved. Also one must keep in mind the fact that the stability line is calculated assuming a plane-wave state with the selected wavelength—the existence of the vortex cores is *not* taken into account.

The sideband instability is of *convective* type: Although the linear instability signifies exponential growth,

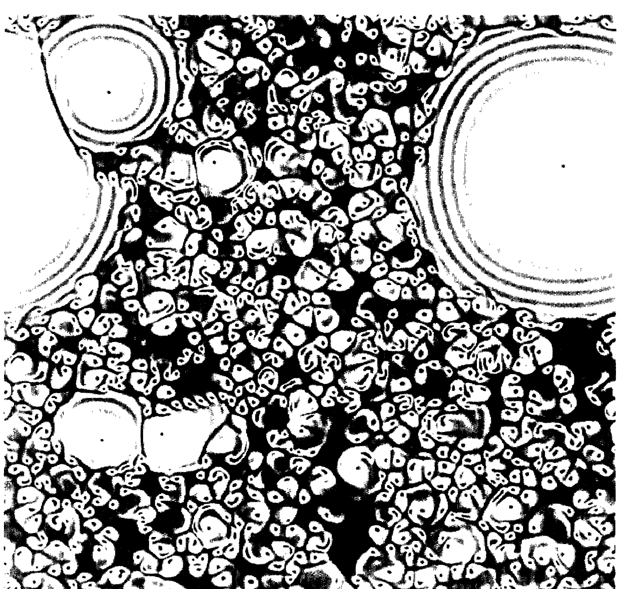

FIG. 3. Nucleating droplets have formed. $\mu=1$, $\alpha=0.8$, $\beta=-1$, and $t=9000$. $|A(\mathbf{x},t)|$ is coded in a grey scale (lattice size 512×512). Vortex centers, where $A=0$, appear as black dots. The nucleating droplets are characterized by a steep increase in absolute amplitude, from zero at the vortex center to a constant value in the vortex-free interior of the droplet. Near the boundary, the outer perturbations give rise to amplitude oscillations.

this growth is only observed in a frame moving with the group velocity. This was noted recently by Aranson *et al.* [10] who further conjectured (with support from simulation) that the onset of turbulence $\alpha=\alpha_0$ (i.e., the point where T diverges) is close to the onset of *absolute* instability, where the exponential instability takes place even in the rest frame. We have checked this conjecture for our coupled-map representation by extending the stability analysis to the complex plane and computing the eigenvalue χ that corresponds to the saddle point. In fact we find that χ passes through 1 very near the onset of turbulence and we now show how this can be used to estimate the transient time T and find its dependence on the parameters.

A nucleating droplet contains only one vortex (spiral wave). A necessary condition for the droplet to form is that this central vortex survives outer turbulent perturbations for a time period Δt large enough to locally stabilize the spiral wave. A perturbation decays like χ^t, so the time needed is $\Delta t\sim|\ln\chi|^{-1}$. If v denotes the (average) turbulent vortex velocity, we are led to the following condition: For a droplet to form, the distance R from the "nucleating" vortex to the closest perturbation center (outer vortex) must be at least

$$R=v\Delta t\sim v|\ln\chi|^{-1}. \tag{5}$$

For fixed β, an expansion to lowest order in α yields $v\sim v_0$ and $\ln\chi\sim\alpha-\alpha_0$, where v_0 is the turbulent velocity at the instability. Thus, by (5),

$$R\sim(\alpha_0-\alpha)^{-1}. \tag{6}$$

The nucleation time T is the time it takes for a nucleating droplet to form. From the considerations above, T is the time we have to wait before a single vortex somewhere becomes separated from all other vortices by a distance larger than R. If we assume that the vortices move randomly, then $T\sim 1/p$, where p is the probability that an area πR^2, that contains a single vortex, exists in one realization of randomly distributed vortices. In each region of size πR^2, the probability p_1 of finding exactly one vortex is given by the binomial form

$$p_1=Nr(1-r)^{N-1},\quad r\equiv n/N, \tag{7}$$

where $n=\rho\pi R^2$, and N is the total number of vortices. Moreover, $1-p=[1-p_1]^{N/n}$. For $1\ll n\ll N$, we have $p\sim Ne^{-n}$, and thus by (6)

$$\ln(T/T_N)\sim(\alpha_0-\alpha)^{-2}, \tag{8}$$

where the time scale T_N decreases with N [11]. We have determined $T(\alpha)$ in our simulations (for $\beta=-1$), and as shown in Fig. 4, the analytic form (8) fits the data very well. The transient time T was computed by determining when the vortex density had decayed by 2 standard deviations from its average value in the metastable turbulent state.

It is remarkable that the transition to turbulence can

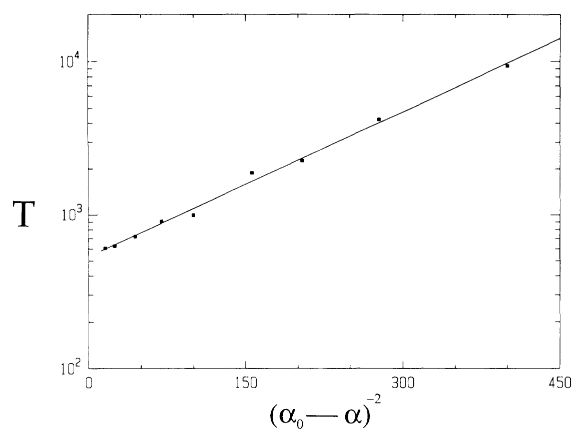

FIG. 4. Semilogarithmic plot of nucleation time T vs $(\alpha_0-\alpha)^{-2}$ ($\mu=1$, $\beta=-1$). T was averaged over five runs, and in each case determined as the difference between the time $t=T_s$ at the density minimum, and the time at which the density had decreased 2 standard deviations (towards the frozen state) from its average value in the metastable turbulent state.

take place both below and above the Benjamin-Feir line as shown in Fig. 1. In a recent study of the one-dimensional complex Ginzburg-Landau equation [12], it was claimed that the transitions in those two cases should be very different, and that it should be possible to see "phase turbulence," i.e., turbulence caused by strong phase fluctuations *without* defects (or with a time-independent defect density), in the region above the Benjamin-Feir line but below the transition to "defect turbulence." We note as a comment that we have *not* seen such states in our simulations.

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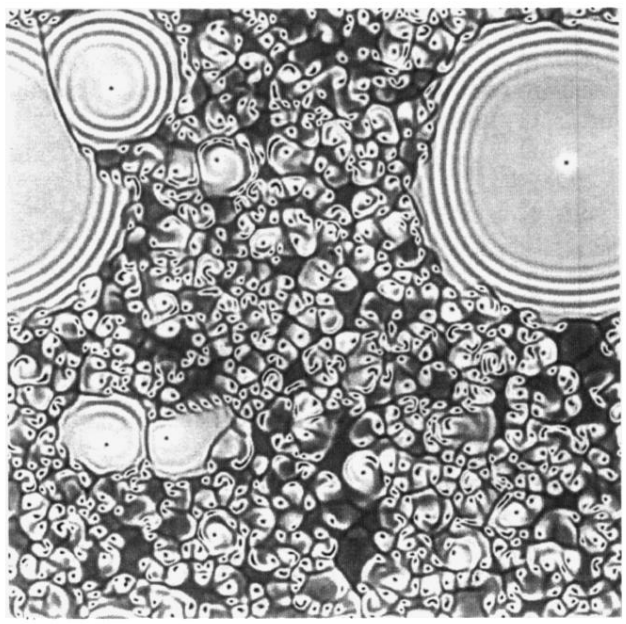

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