Weaving a Classical Metric with Quantum Threads

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Results that illuminate the physical interpretation of states of nonperturbative quantum gravity are obtained using the recently introduced loop variables. It is shown that (i) while local operators such as the metric at a point may not be well defined, there do exist *nonlocal* operators, such as the area of a given two-surface, which can be regulated diffeomorphism invariantly and which are finite *without* renormalization; (ii) there exist quantum states which approximate a given metric at large scales, but such states exhibit a discrete structure at the Planck scale.

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It is by now generally accepted that perturbative approaches to quantum gravity fail because they assume that space-time geometry can be approximated by a smooth continuum at all scales. What is needed are nonperturbative approaches which can *predict*—rather than assume—what the true nature of the microstructure of this geometry is. In such an approach, background fields such as a classical metric or a connection cannot play a fundamental role; quantum theory must be formulated in a diffeomorphism-invariant fashion. An important task in these programs is then to introduce techniques needed to describe geometry and to "explain" from first principles how smooth geometries can arise on macroscopic scales.

Over the past five years, two avenues have been pursued to test if quantum general relativity can exist nonperturbatively. The first is based on numerical simulations [1], while the second is based on canonical quantization [2-6]. This Letter concerns the second approach. While the canonical approach itself was introduced by Dirac in the late 1950s, the recent work departs from the early treatment in two important ways: (i) It is based on a new canonically conjugate pair, the configuration variable being a connection [2,5]; and (ii) it uses a new representation in which quantum states arise as suitable functions on the space of closed loops on a (spatial) 3-manifold [2,3,6]. The new ingredients have led to technical as well as conceptual simplifications which, in turn, have led to a variety of new results. In particular, these methods have opened up bridges between quantum gravity and other areas in mathematics and physics such as knot theory, Chern-Simons theory, and Yang-Mills theory.

The purpose of this Letter is to report on the picture of quantum geometry that arises from the use of the loop variables. To explore the geometry nonperturbatively, we must first introduce operators that carry the metric information and regulate them in such a way that the final operators do *not* depend on any background structure introduced in the regularization. We will show that such operators do exist and that they are finite without renormalization. Using these operators, we seek nonperturbative states which can approximate a given classical geometry up terms $O(l_P/L)$, where l_P is the Planck length and L is a macroscopic length scale, lengths being defined by the given metric. We find that such states do exist but that they exhibit a discrete structure at the Planck scale l_P. Such a result was anticipated on general grounds since the 1930s. Indeed, there exist a number of quantum gravity programs that *begin* by postulating discrete structures at the Planck scale and then attempt to recover from it the known macroscopic physics [7]. The key difference is that, in our approach, discreteness is *arrived at* by combining general relativity with quantum mechanics using loop variables.

In this Letter, we will only sketch the main ideas involved; details will appear elsewhere [8].

Let us begin with the classical phase space. The configuration variable A_a^i is a complex SU(2) connection and its conjugate momentum $\tilde{E}_i^a$, the mathematical analog of the electric field in Yang-Mills theory, is a triad with density weight 1 [5]. (Throughout we will let a, $b, \ldots$ denote the spatial indices and $i,j,\ldots,$ the internal indices. A tilde over a letter will denote a density weight 1.) The first step is the introduction of loop variables [6] which are manifestly SU(2)-gauge-invariant functions on the phase space. The configuration variables are the Wilson loops: Given a closed loop γ on the 3-manifold Σ, we set

$$T[\gamma] = \tfrac{1}{2}\,\mathrm{Tr}\,\mathcal{P}\exp G\oint_\gamma A_a\, dl^a\,, \tag{1}$$

where G is Newton's constant. (Throughout, we use the 2-dimensional representation of the gauge group to evaluate traces.) Variables with momentum dependence are constructed by inserting E_i^a at various points on the loop before taking the trace. Thus, for example, the loop variable quadratic in momenta is given by

$$T^{aa'}[\gamma](y,y') = \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\mathcal{P}\exp G\int_{y'}^{y} A_a\, d\gamma^a\right)\tilde{E}^a(y')\times\left(\mathcal{P}\exp G\int_{y}^{y'} A_a\, d\gamma^a\right)\tilde{E}^{a'}(y)\right], \tag{2}$$

where y and y' are any two points on the loop γ. Note that in the limit when the loop γ shrinks to a single point x, $T^{aa'}[\gamma]$ tends to $-4\tilde{E}_i^a(x)\tilde{E}^{a'i}(x)$, which, when $\tilde{E}_i^a$ is invertible, is related to the metric q_{ab} by $\det q(x) \times q^{aa'}(x) = \tilde{E}_i^a(x)\tilde{E}^{a'i}(x)$. Thus (if the fields A_a and $\tilde{E}^a$ are smooth) one can recover the metric from the loop variable $T^{aa'}$.

In quantum theory, states are represented by suitably regular functions $\Psi[\alpha]$ of loops satisfying certain algebraic conditions and quantum operators corresponding to the loop variables are defined in such a way that the Poisson algebra of the loop variables is mirrored in the commutator algebra in the usual fashion. To define these operators, it is convenient to use the bra-ket notation, $\Psi[\alpha] = \langle\alpha|\Psi\rangle$, and specify the action of operators on bras $\langle\alpha|$ which correspond directly to (certain equivalence classes [9] of) loops. For example, the action of the loop operator $\hat{T}[\gamma]$ is given by

$$\langle\alpha|\circ\hat{T}[\gamma] = \tfrac{1}{2}(\langle\alpha\#\gamma| + \langle\alpha\#\gamma^{-1}|)\,, \tag{3}$$

where $\alpha\#\gamma$ is an "eye glass loop" which is equal to $\alpha\circ\eta\circ\gamma\circ\eta^{-1}$ for an arbitrary segment η joining α and γ and the same segment is used in both terms [10]. Note that, because the connection A_a is complex, the $\hat{T}_\alpha$ are not expected to be self-adjoint, whence there is an intrinsic asymmetry between their action on bras and kets. Similarly, the action of higher-order loop operators such as $\hat{T}^{aa'}[\gamma](y,y')$ also involves just gluing, breaking, and rerouting of loops.

It is tempting to try to define the local metric operator as the limit of $\hat{T}^{ab}$. However, the resulting operator has to be regulated and then renormalized—it involves products of $\tilde{E}_i^a$ and $\tilde{E}_i^b$ evaluated at the *same* point—and, because of the density weights involved, the renormalized operator carries an imprint of the background structure used in this procedure. This is because the renormalization procedure changes the density weight as it replaces a product of delta functions by a single delta function and, in the absence of a metric, delta functions are densities. (In Minkowskian field theories, there *is* a preferred background metric and the only ambiguity in defining analogous operators is that of a multiplicative renormalization *constant*.) This appears to be a general feature of diffeomorphism-invariant theories and it obstructs the introduction of meaningful *local* operator-valued distributions carrying geometric information.

Fortunately, however, there do exist nonlocal operators carrying the same information. We now sketch the construction of two of these.

Note first that, given a smooth 1-form ω_a on Σ, we can define a function $Q[\omega]$ on the classical phase space which carries the metric information,

$$Q[\omega] := \int_\Sigma d^3x\,(\tilde{E}_i^a\omega_a\tilde{E}^{a'i}\omega_{a'})^{1/2}\,, \tag{4}$$

where the integral on the right is well defined because the integrand is a density of weight 1. When the triads are smooth, we can reconstruct the density-weighted inverse metric from the knowledge of $Q[\omega]$ (for all ω). In terms of the classical loop variable $T^{aa'}$, this function can be reexpressed as

$$Q[\omega] = \lim_{\epsilon\to 0}\int d^3x\left[\int d^3y\int d^3y' f_\epsilon(x,y)f_\epsilon(x,y')(-\tfrac{1}{4})T^{aa'}[\gamma_{y,y'}](y,y')\omega_a(y)\omega_{a'}(y')\right]^{1/2}, \tag{5}$$

where $f_\epsilon(x,y)$ is a smearing function, a density of weight 1 in x, which tends to $\delta^3(x,y)$ as ϵ tends to zero and where $\gamma_{y,y'}$ is any smooth loop that passes through points y and y', such that it shrinks smoothly to a point as $y'\to y$. Expression (5) of $Q[\omega]$ is well suited for translation to quantum theory: We can define the quantum operator $\hat{Q}[\omega]$ simply by replacing $T^{aa'}$ in (5) by the loop operator $\hat{T}^{aa'}$ and taking the limit $\epsilon\to 0$ in the action of the operator on states. The resulting operator is well defined—it carries no memory of the additional structure used in the definition of the smearing functions—and finite without any renormalization [11]. The resulting action of the operator is quite simple. If α is a nonintersecting loop [12], it is

$$\langle\alpha|\circ\hat{Q}[\omega] = \frac{l_P^2}{2}\oint_\alpha ds\,|\dot{\alpha}^a\omega_a(\alpha(s))|\langle\alpha|\,. \tag{6}$$

Thus, on nonintersecting loops, the operator acts simply by multiplication. Hence the loop representation is well suited to find states in which the 3-geometry—rather than its time evolution—is sharp.

The second class of operators corresponds to the area of 2-surfaces. Note first that, given a smooth 2-surface S in Σ, its area $\mathcal{A}_S$ is a function on the classical phase space. We first express it using the classical loop variables. Let us divide the surface S into a large number N of area elements S_I, $I=1,2,\ldots,N$, and set $\mathcal{A}_I^{\rm appr}$ to be

$$\mathcal{A}_I^{\rm appr} = \left[\int_{S_I} d^2S^{bc}(x)\epsilon_{abc}\int_{S_I} d^2S'^{b'c'}(x')\epsilon_{a'b'c'}(-\tfrac{1}{4})T^{aa'}[\gamma_{x,x'}](x,x')\right]^{1/2}, \tag{7}$$

where ϵ_{abc} is the (metric independent) Levi-Civita density of weight -1. Since $T^{aa'}$ approximates $(\det q)q^{ab}$ for smooth metrics, $\mathcal{A}_I^{\rm appr}$ approximates the area function (on the phase space) defined by the surface elements S_I, the approximation becoming better as S_I, and hence loops $\gamma_{x,x'}$, shrink. Therefore, the total area $\mathcal{A}_S$ associated with S is given by

$$\mathcal{A}_S = \lim_{N\to\infty}\sum_I \mathcal{A}_I^{\rm appr}\,. \tag{8}$$

To obtain the quantum operator $\hat{\mathcal{A}}_S$, we simply replace $T^{aa'}$ by the quantum loop operator $\hat{T}^{aa'}$. This somewhat indirect procedure is necessary because there is no well-defined operator that represents the metric or its area element *at a point*. Again, the operator $\hat{\mathcal{A}}_S$ is finite and its action is simple when evaluated on a nonintersecting [12] loop α:

$$\langle\alpha|\circ\hat{\mathcal{A}}_S=(l_P^2/2)I(S,\alpha)\langle\alpha|\,, \tag{9}$$

where $I(S,\alpha)$ is simply the unoriented intersection number between the 2-surface S and the loop α [4]. Thus, in essence, a loop α contributes half a Planck unit of area to any surface it intersects. The area operator also acts simply by multiplication on nonintersecting loops.

Because of the simple form of operators $\hat{Q}[\omega]$ and $\hat{\mathcal{A}}_S$, a large set of simultaneous eigenbras can be immediately constructed. There is one, $\langle\gamma|$, associated to every nonintersecting loop γ [13]. Note that the corresponding eigenvalues of area are then *quantized* in integral multiples of $l_P^2/2$. There are also eigenstates associated with intersecting loops; these are discussed in [4,7].

Let us now turn to the second of our main results. The goal here is to introduce loop states which approximate a given 3-metric h_{ab} on Σ on scales L large compared to l_P. (Note that the large scale limit is equivalent to the semiclassical limit since, in source-free, nonperturbative quantum general relativity, $\hbar$ and G always occur in the combination $\hbar G=l_P^2$.) The basic idea is to weave the classical metric out of quantum loops by spacing them so that on an average precisely one line crosses every surface element whose area, *as measured by the given* h_{ab}, is one Planck unit. Such loop states will be called *weaves*. Given a weave, one can obtain others by, e.g., adding small fluctuations.

We now present a concrete example of such a state which approximates a *flat* metric h_{ab}. Using this metric, let us introduce a random distribution of points on $\Sigma=R^3$ with density n (so that in any given volume V there are $nV[1+O(1/\sqrt{nV})]$ points). Center a circle of radius $a=(1/n)^{1/3}$ at each of these points, with a random orientation. We assume that $a\ll L$, so that there is a large number of (nonintersecting but, generically, linked) loops in a macroscopic volume L^3. Denote the collection of the circles by Δ. Because of the identities [6,9] satisfied by the loop states, multiloops are equivalent to single loops, whence there is a well-defined bra $\langle\Delta|$. This is, as we now indicate, a weave state with the required properties. To see if it reproduces on a scale $L\gg l_P$ the geometry determined by the classical metric h_{ab}, let us introduce a 1-form ω_a which is slowly varying on the scale L and compare the value $Q[\omega](h)$ of the classical $Q[\omega]$ evaluated at the metric h_{ab}, with the action of the quantum operator $\hat{Q}[\omega]$ on $\langle\Delta|$. A detailed calculation yields

$$\langle\Delta|\circ\hat{Q}[\omega]=\left[\frac{\pi}{2}\left(\frac{l_P}{a}\right)^2 Q[w](h)+O\left(\frac{a}{L}\right)\right]\langle\Delta|\,. \tag{10}$$

Thus, $\langle\Delta|$ is an eigenstate of $\hat{Q}[\omega]$ and the corresponding eigenvalue is closely related to $Q[\omega](h)$. However, even to the leading order, the two are unequal *unless* the average distance a between the centers of loops *equals* $\sqrt{\pi/2}l_P$. More precisely, writing the leading coefficient as $\frac{1}{4}(2\pi a/l_P)nl_P^3$ we see that, to approximate h_{ab}, Δ should contain, on an average, one-fourth Planck length of curve per Planck volume, where lengths and volumes are measured using h_{ab}.

The situation is the same for the area operators $\hat{\mathcal{A}}_S$. Let S be a 2-surface whose extrinsic curvature varies slowly on a scale $L\gg l_P$. The state $\langle\Delta|$ is an eigenvector of $\hat{\mathcal{A}}_S$ with eigenvalue equal [up to $O(l_P^2/\mathcal{A}_S(h))$] to the area $\mathcal{A}_S(h)$ assigned to S by h_{ab} when the mean separation a between loops satisfies precisely the condition stated above.

Thus, the requirement that $\langle\Delta|$ should approximate the classical metric h_{ab} on *large scales* L tells us something nontrivial about the *short-distance structure* of the multiloop Δ: a is fixed to be $\sqrt{\pi/2}$ times the Planck length. Now, naively, one might have expected that the best approximation to the classical metric would occur in the continuum limit in which the separation a between loops goes to zero. However, this limit is inappropriate because, as a tends to zero, the eigenvalues of $\hat{Q}[\omega]$ and $\mathcal{A}_S$ actually diverge. The reason is that the factors of the Planck length in (6) and (9) force each loop in the weave to contribute a Planck unit to the two geometrical observables. Next, note the structure of the argument: We being with a classical metric, use it to define the scale L, the notion of "slowly varying," as well as the structure of Δ, and find that the mean separation a between the loops is forced to be $\sqrt{2\pi}l_P$, as measured by h_{ab}.

Finally, the above construction can be extended [7] to *curved* metrics which are slowly varying with respect to the flat metric h_{ab}, used above: Given a slowly varying tensor field $t_a{}^b$, the metric $q_{ab}=t_a{}^c t_b{}^d h_{cd}$ can be approximated by a weave Δ_t constructed by "deforming" Δ using $t_a{}^b$.

We conclude with three remarks.

(1) The weave Δ approximates h_{ab} only when we use smearing fields ω_a (and 2-surfaces S) which are slowly varying with respect to h_{ab}. Therefore, in principle it is possible that the same weave Δ can approximate another metric, h'_{ab}, which fails to be slowly varying with respect to h_{ab} and therefore defines a *distinct* class of smearing fields, which are now slowly varying with respect to h'_{ab}. Whether this can occur is being investigated. In any case, it is clear that Δ will *not* approximate h'_{ab} if h'_{ab} is slowly varying with respect to h_{ab}.

(2) The relation between the exact theory and the linearized theory is being investigated [14]. There are preliminary indications that it is possible to reconstruct the (low frequency) graviton states [15] from the states of the exact theory which are "near" $\langle\Delta|$. In this connection, note also that, as it is an eigenstate of the metric,

$\langle\Delta|$ is not a candidate for the "vacuum" of the theory. However, candidates for the vacuum may be constructed by dressing $\langle\Delta|$ with an appropriate distribution of loops corresponding to the virtual gravitons.

(3) The main results presented in this Letter can be obtained also in the connection representation [2,5,16], in which case the weave state is represented by the functional $\Psi_\Delta[A]=\mathrm{Tr}\mathcal{P}\exp(G\oint_\Delta A_a\, d\Delta^a)$. Thus, it is the loop operators (rather than the loop states) that are essential to the argument; they provide us with a regularization procedure that respects diffeomorphism invariance. However, for a nonperturbative treatment of dynamics —i.e., for constructing physical states which are annihilated by the constraints—the use of the loop representation seems unavoidable since, at present, solutions to all quantum constraints are known only in this representation. The relation between the knot classes, which arise as solutions, and 3-geometries—i.e., diffeomorphism equivalence classes of 3-metrics—is being investigated.

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[1] B. S. DeWitt, E. Myers, R. Harrington, and A. Kapulkin, Nucl. Phys. B (Proc. Suppl.) **20**, 744 (1991); M. E. Agishtein and A. A. Migdal, Princeton University Report No. PUPT-1272, 1991 (to be published).

[2] A. Ashtekar, *Non-Perturbative Canonical Quantum Gravity* (notes prepared in collaboration with R. Tate) (World Scientific, Singapore, 1991).

[3] C. Rovelli, Classical Quantum Gravity **8**, 1613 (1991).

[4] L. Smolin, Syracuse University Report No. SU-GP-92/2-2 [in Proceedings of the 1991 GIFT International Seminar on Theoretical Physics (World Scientific, Singapore, to be published), hepth No. 9202022].

[5] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987).

[6] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B133**, 80 (1990).

[7] See, for example, D. Finkelstein, Phys. Rev. **184**, 1261 (1969); R. Penrose, in *Quantum Theory and Beyond*, edited by T. Bastin (Cambridge Univ. Press, New York, 1971); G. 't Hooft, in *Recent Developments in Gravitation*, edited by M. Levy and S. Deser (Plenum, New York, 1979); L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Phys. Rev. Lett. **60**, 655 (1988); J. Barbour and L. Smolin, Syracuse University Report No. SU-GP-92/2-4, 1992 (to be published).

[8] A. Ashtekar, C. Rovelli, and L. Smolin (to be published); preliminary versions of some of these results are reported in [3,4].

[9] A. Ashtekar and C. J. Isham, Classical Quantum Gravity (to be published).

[10] This definition is equivalent to the one given in [2–4,6] in which the state space is extended to functions on multiloops and $\hat{T}[\gamma]\circ\Psi[\alpha]=\Psi[\alpha\cup\gamma]$.

[11] In fact, the finiteness of these operators is closely related to their background independence. Basically, a dependence on a power of the regulator must come together with background dependence, because the regulator scale is defined with respect to the background metric introduced in regularization. It is tempting to conjecture that any operator that is regulated in such a way that no dependence on the background survives in the limit in which the regulator is removed must be finite in this limit.

[12] The action of the operators on intersecting loops is known explicitly [4,7], but is slightly more complicated.

[13] Such states are not normalizable in Minkowskian quantum field theories based on Fock inner products. However, in the absence of a background metric, the Fock representation cannot be defined. Quantum field theories on manifolds without background metrics may be nonetheless constructed using representations of the sort introduced in [9] and in such representations bras $\langle\gamma|$ *are* normalizable.

[14] J. Iwasaki and C. Rovelli, Pittsburgh University report, 1992 (to be published); see also J. Zegwaard, Utrecht Report No. THU-91/13, 1991 (to be published).

[15] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. D **44**, 1740 (1991).

[16] T. Jacobson and L. Smolin, Nucl. Phys. **B299**, 295 (1988).