Construction of Inhomogeneous Universes Which Are Friedmann-Lemaître-Robertson-Walker on Average

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(Received 29 May 1992)

The understanding of our Universe is based on the working hypothesis that the homogeneous and isotropic models give a successful description on a very large scale, despite the nonlinear inhomogeneity of the matter distribution in the present Universe. We consider the compatibility problem between the overall homogeneity and isotropy and the local inhomogeneity. A scheme to construct inhomogeneous irrotational dust universes which are homogeneous and isotropic on average is shown in the framework of general relativity; they represent "relativistic pancake solutions" analogous to those in Newtonian cosmology.

PACS numbers: 98.80.Dr

The remarkable isotropy of the cosmic microwave background now strongly supports the idea that the Universe is homogeneous and isotropic in some averaged sense and well described by the Friedmann-Lemaître-Robertson-Walker (FLRW) models. The real Universe is, however, neither homogeneous nor isotropic, and the density contrast is larger than unity up to the scale of superclusters. Although the FLRW models provide a successful description of the averaged overall behavior, the smoothing procedure, i.e., the transition from a locally clumpy universe to an averaged FLRW universe, is still not understood in detail [1], mainly because of the nonlinearity of the Einstein equations. The pioneering work in understanding the smoothing process is due to Futamase [2–4]. In his approximation scheme the spatial averaging is introduced to derive the averaged behavior of the spacetime, but the ansatz for the metric is made such that the deviations from the FLRW models are small. In this Letter spatial averaging is also introduced, but we shall proceed differently to construct clumpy universes whose overall behavior is FLRW-like on average.

We consider irrotational dust with density ρ and four-velocity u^μ. In comoving coordinates with $u^\mu=(1,0,0,0)$ and $g_{0i}=0$, $u_\mu T^{\mu\nu}{}_{;\nu}=0$ and the Einstein equations read

$$\dot\rho+\rho u^i{}_i=0\,, \tag{1}$$

$$\tfrac{1}{2}\{{}^{(3)}R+(u^i{}_i)^2-u^i{}_j u^j{}_i\}=8\pi G\rho\,, \tag{2}$$

$$u^i{}_{j|i}-u^i{}_{i|j}=0\,, \tag{3}$$

$$\dot u^i{}_j+u^k{}_k u^i{}_j+{}^{(3)}R^i{}_j-4\pi G\rho\delta^i{}_j=0\,, \tag{4}$$

$$\dot g_{ik}=2g_{ij}u^j{}_k\,, \tag{5}$$

where $\cdot\equiv\partial/\partial t$, $|$ denotes the covariant derivative with respect to g_{ij}, ${}^{(3)}R^i{}_j$ is the Ricci tensor of the three-metric, ${}^{(3)}R={}^{(3)}R^i{}_i$, and $u^i{}_j\equiv u^i{}_{;j}$ is the deformation tensor which describes the change of the relative position X^i between the world lines of neighboring "particles" (say, galaxies) [5], $\dot X^i=u^i{}_jX^j$. ($u^i{}_j$ is also known as the extrinsic curvature of the $t=$const hypersurfaces.)

The standard (but a bit simplified) approach in observational cosmology may be described as follows: (a) Observe the distribution, the masses, and the velocities relative to us of neighboring galaxies; (b) calculate the averaged quantities under the assumption that the relative velocity field is isotropic on average (the Hubble law) and the overall distribution is homogeneous; and (c) compare these mean properties with those of the FLRW models with the same density as that of the total mass of the galaxies distributed uniformly in the observed region. We would be happy if we could get the best-fit parameters of the FLRW models in this way. The discrepancy between the observational data and the properties of the FLRW models, however, usually requires additional matter contents besides that estimated from the visible part of galaxies, such as dark matter or even a cosmological constant.

We will not discuss the observational aspects in detail, but we shall follow this approach to describe the average behavior of an inhomogeneous universe. In the comoving coordinate system that we took, Σ_t, the hypersurface $t=$const is orthogonal to the dust motion. The FLRW models require that the matter distribution be perfectly homogeneous on Σ_t. Therefore, a corresponding mean (or "background") density for an inhomogeneous universe is defined by

$$\rho_b=\langle\rho\rangle\equiv\lim_{V\to\Sigma_t}\frac{1}{\int_V[\det(g_{ij})]^{1/2}d^3x}\int_V\rho[\det(g_{ij})]^{1/2}d^3x\,,\quad V\subset\Sigma_t\,. \tag{6}$$

It is assumed that this limit exists. The scale factor $a(t)$ is defined by

$$\dot\rho_b+3(\dot a/a)\rho_b=0\,. \tag{7}$$

We define the peculiar deformation tensor $V^i{}_j$ by

$$V^i{}_j\equiv u^i{}_j-(\dot a/a)\delta^i{}_j\,, \tag{8}$$

which represents the deviation from the uniform Hubble expansion, and the density contrast $\Delta\equiv(\rho-\rho_b)/\rho$, which is more convenient than the conventional definition $\delta \equiv (\rho - \rho_b)/\rho_b$ [see Eq. (13) and the discussions below]. Using these quantities, Eq. (1) turns into

$$\dot{\Delta} + (1-\Delta) V^i{}_i = 0\,, \tag{9}$$

and Eq. (2) becomes

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_b}{1-\Delta} - \frac{1}{6}{}^{(3)}R - \frac{2}{3}\frac{\dot a}{a}V^i{}_i - \frac{1}{6}\{(V^i{}_i)^2 - V^i{}_j V^j{}_i\}\,. \tag{10}$$

In the absence of inhomogeneities, this equation is simplified to Friedmann's equation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_b - \frac{k}{a^2}\,. \tag{11}$$

It follows that our average quantities ρ_b, a are compatible with a Friedmann "background" model if and only if

$$\frac{8\pi G}{3}\rho_b\frac{\Delta}{1-\Delta} - \frac{1}{6}{}^{(3)}R + \frac{k}{a^2} - \frac{2}{3}\frac{\dot a}{a}V^i{}_i - \frac{1}{6}\{(V^i{}_i)^2 - V^i{}_j V^j{}_i\} = 0\,. \tag{12}$$

If this holds, Eq. (1) and the trace of Eq. (4) give

$$\ddot{\Delta} + 2(\dot a/a)\dot{\Delta} - 4\pi G\rho_b\Delta = -(1-\Delta)\{(V^i{}_i)^2 - V^i{}_j V^j{}_i\}\,. \tag{13}$$

The left-hand side of Eq. (13) is familiar to cosmologists. Its vanishing governs δ in linear perturbation theory [6-9]. The solutions for δ are given in the literature [10]. Equations almost identical to Eqs. (9) and (13) have been derived in Newtonian cosmology in the context of extending the Zel'dovich-type approximation (pancake solutions) [11-14]. ($V^i{}_j$ correspond to the spatial gradient of the peculiar velocity, $v^i{}_{,j}$.)

The close resemblance to the treatment in linear theory leads us to consider the case

$$(V^i{}_i)^2 - V^i{}_j V^j{}_i = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) = 0\,, \tag{14}$$

where λ_i ($i=1,2,3$) are the eigenvalues of $V^i{}_j$. In this case, Δ obeys exactly the same equation as δ, and the solution can be *extrapolated* from the result of the linear perturbation theory [12,13]. A simple example is that two eigenvalues, say, λ_2 and λ_3, vanish. Physically speaking, this means that the collapse is (locally) one dimensional, which is exactly the property of Zel'dovich's solution [11,13]. Therefore, the solution we are looking for here is regarded as a "general relativistic pancake solution." Let us consider the case that the metric is diagonal. Employing the growing and decaying mode solutions $\Delta_+(t)$ and $\Delta_-(t)$, and taking the constraint Eq. (3) into account, the metric can be written in the form

$$ds^2 = -dt^2 + a^2(t)\{Q^2dx^2 + \phi^2(y,z)(dy^2+dz^2)\}\,, \tag{15}$$

where $Q = A(x,y,z) + \beta(x)\Delta_+(t) + \mu(x)\Delta_-(t)$. The collapse occurs in the x direction.

When the two-spaces $t=$const, $x=$const are of constant curvature k, the exact solutions have been found by Szekeres [15,16]. For presentation's sake, we shall give the expression for $k=0$. (The general cases $k\neq 0$ can also be treated in the same way.) In this case, we can set $\phi(y,z)=1$. The solutions for the density contrast are $\Delta_+(t) = t^{2/3}$, $\Delta_-(t) = t^{-1}$ (see also Refs. [17-19]), and $A(x,y,z) = \frac{5}{9}\beta(x)(y^2+z^2) + \sigma(x)y + \nu(x)z + \omega(x)$. By a suitable transformation, one of the five functions of x, say, $\omega(x)$, can be normalized to 1. Therefore, the other four arbitrary functions characterize the solution:

$$V^i{}_j = \mathrm{diag}\left(\frac{1}{Q}(\beta\dot{\Delta}_+ + \mu\dot{\Delta}_-),\, 0,\, 0\right), \tag{16}$$

$${}^{(3)}R^i{}_j = \mathrm{diag}\left(-\frac{20}{9}\frac{\beta}{a^2Q},\, -\frac{10}{9}\frac{\beta}{a^2Q},\, -\frac{10}{9}\frac{\beta}{a^2Q}\right), \tag{17}$$

$$8\pi G\rho = \frac{4}{3}\frac{A}{a^3Q} = 8\pi G\rho_b\left(1 - \frac{\beta}{Q}\Delta_+(t) - \frac{\mu}{Q}\Delta_-(t)\right). \tag{18}$$

The only property we shall assume for the functions is that the mean values of $\beta(x)$ and $\mu(x)$ vanish, i.e., $\bar f(x) \equiv \int_{V_x} f(x)dx = 0$ for $f(x) = \beta(x)$ and $\mu(x)$. This leads to the interesting consequence that the average behavior of the spacetime is FLRW-like:

$$\langle u^i{}_j\rangle = (\dot a/a)\delta^i{}_j\,,\quad \langle {}^{(3)}R^i{}_j\rangle = 0\,,\quad 8\pi G\langle\rho\rangle = 8\pi G\rho_b = 3(\dot a/a)^2 \tag{19}$$

with $a = t^{2/3}$.

We have not assumed that the deviations from a FLRW model are small [2-4] to acquire the FLRW-like behavior on average. It should also be noted that the relativistic extension of the pancake solution does not necessarily require the axially symmetric form for the metric [20]. Such restrictive assumptions might lead to the exclusion of a variety of relevant solutions. The description based on the deformation tensor can give another possibility to obtain solutions to describe more realistic situations. Based on the solutions, it should also be possible to formulate a relativistic version of the Zel'dovich approximation, which has been used successfully to handle the evolution of the large-scale structure in Newtonian cosmology. A more detailed discussion will be published elsewhere.

I would like to thank G. Börner, J. Ehlers, and T. Futamase for valuable remarks on the manuscript, K. Tomita and A. Krasiński for useful information on references, and S. Bildhauer and T. Buchert for stimulating discussions and comments. I also thank the Alexander von

Humboldt-Stiftung for financial support during the stay at Max-Planck-Institut für Astrophysik.

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[1] G. F. R. Ellis, in *General Relativity and Gravitation*, edited by B. Bertotti *et al.* (Reidel, Dordrecht, 1984).
[2] T. Futamase, Phys. Rev. Lett. **61**, 2175 (1988).
[3] T. Futamase, Mon. Not. R. Astron. Soc. **237**, 187 (1989).
[4] T. Futamase, Prog. Theor. Phys. **86**, 389 (1991).
[5] G. F. R. Ellis, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic, London, 1971).
[6] W. B. Bonnor, Mon. Not. R. Astron. Soc. **117**, 104 (1957).
[7] P. J. E. Peebles, *The Large Scale Structure of the Universe* (Princeton Univ. Press, Princeton, 1980).
[8] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[9] G. F. R. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989).
[10] For example, D. Heath, Mon. Not. R. Astron. Soc. **179**, 351 (1977).
[11] Ya. B. Zel'dovich, Astron. Astrophys. **5**, 84 (1970).
[12] P. J. E. Peebles, Astrophys. J. **317**, 576 (1987).
[13] T. Buchert, Astron. Astrophys. **223**, 9 (1989).
[14] T. Buchert, Mon. Not. R. Astron. Soc. **254**, 729 (1992).
[15] P. Szekeres, Commun. Math. Phys. **41**, 55 (1975).
[16] W. B. Bonnor and N. Tomimura, Mon. Not. R. Astron. Soc. **175**, 85 (1976).
[17] S. W. Goode and J. Wainwright, Phys. Rev. D **26**, 3315 (1982).
[18] S. W. Goode and J. Wainwright, Mon. Not. R. Astron. Soc. **198**, 83 (1982).
[19] S. W. Goode, Classical Quantum Gravity **3**, 1247 (1986).
[20] K. Tomita, Prog. Theor. Phys. **54**, 730 (1975).