Gauge-Invariant Formulations of Lineal Gravities

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It is shown that the currently studied "string-inspired" model for gravity on a line can be formulated as a gauge-invariant theory based on the Poincaré group with central extension—a formulation that complements and simplifies Verlinde's construction based on the unextended Poincaré group.

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Lineal gravities, i.e., Einstein-type theories in $(1+1)$-dimensional space-time, provide a setting for studying nonunderstood issues of gravitation. The simplifications achieved by the drastic dimensional reduction are not devoid of interest, provided dynamical equations are not based on the Einstein tensor $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$, which vanishes identically in two dimensions. Several years ago, a class of theories based on the Riemann scalar R was proposed, but even the simplest of these [1], in which scalar curvature is equated to a cosmological constant Λ,

$$R-\Lambda=0, \tag{1}$$

requires an additional, nongeometrical field in an action formulation: Equation (1) follows from the action

$$I_1=\int d^2x\sqrt{-g}\,\eta(R-\Lambda), \tag{2}$$

where η is an invariant world scalar, which acts as a Lagrange multiplier enforcing (1). Of course, once the additional scalar field has been introduced, one may consider various generalizations and modifications of (1),(2) with alternative dynamics for R and η [2].

The model (1),(2) has two distinctive features. It can be obtained by dimensional reduction from Einstein theory in three space-time dimensions [1]. Moreover, of particular interest for us here, it possesses a gauge-theoretical formulation, given by several people [3]. To this end one uses the de Sitter or anti-de Sitter group with Lorentz generator J and translation generators P_a satisfying the SO(2,1) algebra (for $\Lambda\neq 0$),

$$[P_a,J]=\epsilon_a{}^bP_b,\quad [P_a,P_b]=-\tfrac{1}{2}\Lambda\epsilon_{ab}J. \tag{3}$$

[The tangent-space indices $(a,b,\ldots)$ are raised and lowered with the flat-space metric tensor $h_{ab}=\mathrm{diag}(1,-1)$ and $\epsilon^{01}=1$.] In the usual way, the gauge connection 1-form A is expanded in terms of the generators,

$$A=e^aP_a+\omega J, \tag{4}$$

where e^a_μ is the *zweibein* and ω_μ is the spin connection. The curvature 2-form

$$F=dA+A^2 \tag{5}$$

becomes

$$F=f^aP_a+fJ=(De)^aP_a+(d\omega-\tfrac{1}{4}\Lambda e^a\epsilon_{ab}e^b)J, \tag{6}$$

$$(De)^a\equiv de^a+\epsilon^a{}_b\omega e^b. \tag{7}$$

The three field strengths $F^A=(f^a,f)$ transform covariantly according to the three-dimensional adjoint representation. Therefore the Lagrange density,

$$\mathcal{L}_1'=\sum_{A=0}^{2}\eta_AF^A=\eta_a(De)^a+\eta_2(d\omega-\tfrac{1}{4}\Lambda e^a\epsilon_{ab}e^b), \tag{8}$$

$$\eta_A=(\eta_a,\eta_2),$$

is gauge invariant when the Lagrange multiplier triplet η_A is taken to transform by the coadjoint representation. The equation obtained by varying η_a allows evaluation of the spin connection in terms of the *zweibein*,

$$\omega=e^a(h_{ab}\epsilon^{\mu\nu}\partial_\mu e^b_\nu)/\det e, \tag{9}$$

and the equation that follows upon variation of η_2 regains (1). Finally it is noted that a nondegenerate Killing metric is available because the relevant group is semisimple for $\Lambda\neq 0$.

Recently, Verlinde [4] as well as Callan, Giddings, Harvey, and Strominger [5] have introduced a similar model, which is "string inspired." The action

$$I_2=\int d^2x\sqrt{-g}\,(\eta R-\Lambda) \tag{10}$$

differs from (2) in that the Lagrange multiplier is absent from the cosmological constant [6]. The equation of motion from varying η,

$$R=0, \tag{11}$$

shows that the metric is flat, $g_{\mu\nu}=h_{\mu\nu}$, while varying $g_{\mu\nu}$ gives, with the help of (11),

$$\partial_\mu\partial_\nu\eta=\tfrac{1}{2}\Lambda h_{\mu\nu}. \tag{12}$$

Thus

$$-2\eta=M-\Lambda(x^+-x_0^+)(x^--x_0^-),\quad x^\pm=\frac{1}{\sqrt{2}}(t\pm x), \tag{13}$$

with $x_0^\pm$ and M being integration constants. Interest in the model derives from the above "black-hole" solution [7] with mass M [in terms of the "physical" metric $g_{\mu\nu}/(-2\eta)$] whose quantum mechanical analysis may shed light on various quantum gravity puzzles [4,5,8].

Here we address the problem of a gauge theoretical formulation for I_2. A discussion has already been given by Verlinde [4], based on the nonsemisimple Poincaré group, with the algebra

$$[P_a,J]=\epsilon_a{}^bP_b,\quad [P_a,P_b]=0\,, \tag{14}$$

which is the $\Lambda\to 0$ contraction of (3). However, there are various unexpected features in his formulation: The transformation law for the Lagrange multipliers is an unfamiliar affine expression; the Lagrange density is not invariant but changes by a total derivative—see below. After reviewing Verlinde's approach, we show that an *invariant* Lagrange density with a conventional coadjoint transformation for the Lagrange multipliers can be given, provided one uses a *centrally extended* Poincaré algebra, which is an unconventional contraction of (3).

Following Verlinde, the connection and curvature are defined as in (4)-(7), but owing to the vanishing of the momentum commutator, the curvature becomes the $\Lambda=0$ limit of (6):

$$F=f^aP_a+fJ=(De)^aP_a+d\omega J\,. \tag{15}$$

Infinitesimal gauge transformation rules,

$$\delta A=d\Theta+[A,\Theta]\,, \tag{16}$$

where the gauge generator Θ,

$$\Theta=\theta^aP_a+\alpha J\,, \tag{17}$$

can be deduced from (14) and (16) to be

$$\delta e^a=-\alpha\epsilon^a{}_be^b+\epsilon^a{}_b\theta^b\omega+d\theta^a\,,\quad \delta\omega=d\alpha\,. \tag{18}$$

In finite form they read

$$\begin{aligned}&e^a\to\bar e^a=(\mathcal{M}^{-1})^a{}_b(e^b+\epsilon^b{}_c\theta^c\omega+d\theta^b)\,,\\&\omega\to\bar\omega=\omega+d\alpha\,,\end{aligned} \tag{19}$$

where $\mathcal{M}$ is a finite Lorentz transformation:

$$\mathcal{M}^a{}_b=\delta^a{}_b\cosh\alpha+\epsilon^a{}_b\sinh\alpha\,. \tag{20}$$

The curvature triplet F^A transforms according to the three-dimensional adjoint representation of the Poincaré group, viz., as in (19) but without the differentials:

$$f^a\to\bar f^a=(\mathcal{M}^{-1})^a{}_b(f^b+\epsilon^b{}_c\theta^cf)\,,\quad f\to\bar f=f\,, \tag{21a}$$

or

$$\begin{aligned}&F^A\to\bar F^A=\sum_{B=0}^{2}(U^{-1})^A{}_BF^B\,,\\&U=\begin{pmatrix}\mathcal{M}^a{}_b & -\epsilon^a{}_c\theta^c\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{21b}$$

The Lagrange density is now taken as

$$\begin{aligned}\mathcal{L}_2'&=\sum_{A=0}^{2}\eta_AF^A+\tfrac12\Lambda e^a\epsilon_{ab}e^b\\&=\eta_a(De)^a+\eta_2d\omega+\tfrac12\Lambda e^a\epsilon_{ab}e^b\,.\end{aligned} \tag{22}$$

The equations of motion that follow from $\mathcal{L}_2'$ are equivalent to (11)-(13) (with $\eta_2=-2\eta$). However, the transformation properties of $\mathcal{L}_2'$ under Poincaré gauge transformations are obscure. Following Verlinde, we can check that the following infinitesimal rules for $\delta\eta_A$,

$$\delta\eta_a=\eta_b\epsilon^b{}_a\alpha-\Lambda\epsilon_{ab}\theta^b,\quad \delta\eta_2=-\eta_a\epsilon^a{}_b\theta^b\,, \tag{23}$$

together with (18) change $\mathcal{L}_2'$ by a total derivative. But the affine shift in $\delta\eta_a$, proportional to Λ, is unfamiliar. In the finite version of (23),

$$\begin{aligned}&\eta_a\to\bar\eta_a=(\eta_b-\Lambda\epsilon_{bc}\theta^c)\mathcal{M}^b{}_a\,,\\&\eta_2\to\bar\eta_2=\eta_2-\eta_a\epsilon^a{}_b\theta^b-\tfrac12\Lambda\theta^2\,,\end{aligned} \tag{24}$$

the homogeneous part is a coadjoint transformation ($\bar\eta=\eta U$), while the shift, proportional to Λ, is needed to compensate for the gauge noninvariance of the cosmological constant in (22), and $\mathcal{L}_2'$ changes as

$$\mathcal{L}_2'\to\bar{\mathcal{L}}_2'=\mathcal{L}_2'+\Lambda d(\theta^a\epsilon_{ab}e^b+\tfrac12\theta^2\omega-\tfrac12 d\theta^a\epsilon_{ab}\theta^b)\,. \tag{25}$$

Upon integration the boundary contributions involving e^a and ω may be dropped with the hypothesis that the dynamical variables vanish on the boundary. But because gauge parameters need not vanish, the last term can survive, even though it is a total derivative:

$$I_2'\to\bar I_2'=I_2'-\Lambda\int d^2x(\det\partial\theta^a/\partial x^\mu)\,. \tag{26}$$

All the awkward features of the above formulation disappear if the gauge theory is based on a *central extension* of the Poincaré algebra. We therefore postulate instead of (3),(14)

$$[P_a,J]=\epsilon_a{}^bP_b,\quad [P_a,P_b]=\epsilon_{ab}\tfrac12 i\Lambda I\,, \tag{27}$$

thereby adding the central element I to the generators and effecting a magneticlike modification of the translation algebra [9]. Consequently the connection and curvature now become

$$\begin{aligned}&A=e^aP_a+\omega J+a\tfrac12 i\Lambda I\,,\\&F=dA+A^2=f^aP_a+fJ+g\tfrac12 i\Lambda I\\&\qquad=(De)^aP_a+d\omega J+(da+\tfrac12 e^a\epsilon_{ab}e^b)\tfrac12 i\Lambda I\,,\end{aligned} \tag{28}$$

and the finite gauge transformations with gauge generator

$$\Theta=\theta^aP_a+\alpha J+\beta\tfrac12 i\Lambda I \tag{29}$$

read

$$e^a \to \bar{e}^a = (\mathcal{M}^{-1})^a{}_b (e^b + \epsilon^b{}_c \theta^c \omega + d\theta^b)\,,$$
$$\omega \to \bar{\omega} = \omega + da\,, \tag{30}$$
$$a \to \bar{a} = a - \theta^a \epsilon_{ab} e^b - \tfrac{1}{2}\theta^2 \omega + d\beta + \tfrac{1}{2} d\theta^a \epsilon_{ab} \theta^b\,.$$

The multiplet of curvatures transforms by the adjoint representation of the extended group,

$$f^a \to \bar{f}^a = (\mathcal{M}^{-1})^a{}_b (f^b + \epsilon^b{}_c \theta^c f)\,,$$
$$f \to \bar{f} = f\,, \tag{31a}$$
$$g \to \bar{g} = g - \theta^a \epsilon_{ab} f^b - \tfrac{1}{2}\theta^2 f\,,$$

or

$$F^A \to \bar{F}^A = \sum_{B=0}^{3} (U^{-1})^A{}_B F^B\,,$$
$$U = \begin{pmatrix} \mathcal{M}^a{}_b & -\epsilon^a{}_c \theta^c & 0 \\ 0 & 1 & 0 \\ \theta^c \epsilon_{cd} \mathcal{M}^d{}_b & -\theta^2/2 & 1 \end{pmatrix}. \tag{31b}$$

Note that in the above realization of the gauge action on F, the extension is not visible; I is represented by **0**.

An invariant Lagrange density is simply constructed with an extended multiplet of Lagrange multipliers,

$$\mathcal{L}_2'' = \sum_{A=0}^{3} \eta_A F^A = \eta_a (De)^a + \eta_2 d\omega + \eta_3 (da + \tfrac{1}{2} e^a \epsilon_{ab} e^b)\,,$$
$$\eta_A = (\eta_a, \eta_2, \eta_3)\,, \tag{32}$$

which obey the conventional coadjoint transformation law,

$$\eta_A \to \bar{\eta}_A = \sum_{B=0}^{3} \eta_B U^B{}_A\,, \tag{33a}$$

or

$$\eta_a \to \bar{\eta}_a = (\eta_b - \eta_3 \epsilon_{bc} \theta^c) \mathcal{M}^b{}_a\,,$$
$$\eta_2 \to \bar{\eta}_2 = \eta_2 - \eta_a \epsilon^a{}_b \theta^b - \tfrac{1}{2}\eta_3 \theta^2\,, \tag{33b}$$
$$\eta_3 \to \bar{\eta}_3 = \eta_3\,.$$

Verlinde's affine transformations for η_a, η_2 are now linear in η_3, which is invariant.

Of course the equations of motion for $\mathcal{L}_2''$ are equivalent to those for $\mathcal{L}_2'$, because variation of a in (32) gives

$$d\eta_3 = 0\,. \tag{34}$$

Therefore η_3 is constant, set equal to Λ, in which case $\mathcal{L}_2''$ differs from $\mathcal{L}_2'$ by the total derivative Λda; but it is the presence of this term that renders $\mathcal{L}_2''$ invariant in contrast to $\mathcal{L}_2'$. (Clearly the solution $\eta_3 = 0$ gives an unextended Poincaré gauge theory with vanishing cosmological constant.) Note that varying the additional Lagrange multiplier η_3 provides one more equation of motion,

$$da + \tfrac{1}{2} e^a \epsilon_{ab} e^b = 0\,, \tag{35}$$

which can be always solved, at least locally, because the second term is a closed 2-form.

The extended algebra, in the representation that we are using, possesses a nonsingular Killing metric, $h_{AB} = \sum_{CD} U^C{}_A h_{CD} U^D{}_B$, which is unavailable without the extension:

$$h_{AB} = \begin{pmatrix} h_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{36}$$

This allows construction of the invariant scalar

$$M = -\frac{1}{2\Lambda} \sum_{A,B=0}^{3} \eta_A h^{AB} \eta_B\,, \tag{37}$$

which is also constant by virtue of the equations of motion and is interpreted as the black-hole mass [4].

In conclusion, we see that the class of lineal gravity theories involving a nongeometric Lagrange multiplier possesses two members that are distinguished from the perspective of gauge invariance: the original model (1),(2) based on the SO(2,1) group and the string-inspired model (10) based on the extended Poincaré group. The extended Poincaré model involves an unconventional contraction of the SO(2,1) model: Owing to the well-known ambiguity of two-dimensional angular momentum, in (3) one may replace J by $J + sI/i$ and Λ by Λ/s, and set s to infinity, thereby arriving at (27). Finally we recall that the de Sitter model (2),(8) can be obtained by dimensional reduction of planar gravity; whether there exists a model in (2+1) dimensions that reduces to the string-inspired lineal theory (10),(32) is under investigation.

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[6] Expression (10) is related to the usual formula

$$I_2 = \int d^2x\sqrt{-g}\, e^{-2\varphi}(R - 4\partial_\mu\varphi\,\partial^\mu\varphi - \Lambda)$$

by rescaling the metric with the "dilaton" field φ, $g_{\mu\nu} \rightarrow e^{2\varphi} g_{\mu\nu}$, and defining $\eta = e^{-2\varphi}$.

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[9] This corresponds to a 2-cocycle in the Poincaré group composition law:

$$G(\mathcal{M}_1, q_1)G(\mathcal{M}_2, q_2) = e^{(i/4)\Lambda q_1^a \epsilon_{ab} (\mathcal{M}_1 q_2)^b} G(\mathcal{M}_1\mathcal{M}_2, q_1 + \mathcal{M}_1 q_2) .$$