Finite-Time Vortex Singularity in a Model of Three-Dimensional Euler Flows

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An analytical model of three-dimensional Euler flows which exhibits a finite-time singularity is given. The singularity in vorticity occurs at a velocity field null (stagnation point) which lies on the line joining two vorticity field nulls. It is shown that the vorticity diverges inversely with time.

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The search for finite-time singularities in incompressible Euler flows has consumed a substantial analytical and computational effort in recent years. For two-dimensional flows which tend to zero at infinity and evolve from smooth initial conditions, there is a smooth classical solution for all times [1-3]. Vieillefosse has proposed an analytical three-dimensional model, neglecting gradients of vorticity and shear in the Euler equations, that exhibits a finite-time singularity [4]. Until 1990, extensive computational studies of three-dimensional flows were not conclusive, despite the fact that the numerical methods employed had attained a high level of sophistication [5-7]. Recently, finite-time singularities have been reported in numerical studies of axisymmetric flows with swirl [8], but doubts have been raised that the growth of vorticity in these studies may be no more than exponential [9]. Kerr has carried out numerical studies of the interaction of antiparallel vortex tubes, and has reported that a special class of initial conditions yields a finite-time singularity in three dimensions [10].

In this paper, we propose an analytical model which yields a finite-time singularity from the three-dimensional Euler equations with smooth initial conditions. We take an initial background flow of the form $v_x(\mathbf{x},t=0)=F(y+z)$, $v_y(\mathbf{x},t=0)=F(z+x)$, and $v_z(\mathbf{x},t=0)=F(x+y)$, where F is a smooth function. Hence, the condition $\nabla\cdot\mathbf{v}=0$ is satisfied identically. We assume that $\mathbf{v}$ has a null (stagnation point) at the origin $x=y=z=0$. Then near the null, $\mathbf{v}=\mathbf{x}\cdot(\nabla\mathbf{v})_0$. The tensor $(\nabla\mathbf{v})_0$, which has zero trace for incompressible flows, has two degenerate eigenvalues $\lambda_-=-F'(0)$ and a third eigenvalue $\lambda_+=2F'(0)$. We require that $F'(0)\equiv v_0/L$ be positive; here v_0 and L are chosen to be positive constants. Then two of the eigenvalues λ_- are negative and the third eigenvalue λ_+ is positive. This null is of type A, according to the classification given in Refs. [11-13]. The two streamlines originating from the null along the eigenvector for λ_+ define the so-called γ_A line which is a one-dimensional unstable manifold. In the vicinity of the null, the eigenvectors for the two negative eigenvalues λ_- lie on a two-dimensional plane, called the Σ_A surface, which is a two-dimensional stable manifold. Near the null, the background flow $\mathbf{v}$ is irrotational.

Superimposed on the initial background flow, we have a smooth incompressible localized flow of the form $u_x(\mathbf{x},t=0)=f(y)$, $u_y(\mathbf{x},t=0)=f(z)$, and $u_z(\mathbf{x},t=0)=f(x)$ with $f(0)=0$. The flow $\mathbf{u}$ also has a null at the origin where $\mathbf{u}=\mathbf{x}\cdot(\nabla\mathbf{u})_0$. The tensor $(\nabla\mathbf{u})_0$ has the eigenvalues $\lambda_0=f'(0)$ and $\lambda_\pm=f'(0)\exp(\pm i2\pi/3)$. We choose $f(x)=u_0(x/a)\exp(-x^2/a^2)$, where u_0 and a are positive parameters and $a\ll L$. This choice is sufficient to localize the flow $\mathbf{u}$ near the origin. Since $f'(0)=u_0/a>0$, the null is of type A_s. The subscript s denotes the spiraling trajectories of the streamlines into the null in the two-dimensional surface Σ_{A_s} containing the eigenvectors for the complex eigenvalues $\lambda_\pm$. This two-dimensional surface is the stable manifold for the null of $\mathbf{u}$, whereas the γ_{A_s} line is the unstable manifold associated with the eigenvector for the positive eigenvalue λ_0. The global geometry of a type A_s null is similar to that of a type A null. The only difference between the two types is the spiraling structure near the null in A_s.

The localized flow $\mathbf{u}$ (as well as the background flow $\mathbf{v}$) obey the symmetry relations $u_x(x,y,z)=u_y(z,x,y)=u_z(y,z,x)$. An advantage of this high degree of symmetry is that the knowledge of a single component of the velocity is sufficient to describe the two other components. We exploit this feature in the analytical calculations that follow.

Unlike the background flow $\mathbf{v}$ which is irrotational near the origin, the localized flow $\mathbf{u}$ produces a vorticity field $\omega_x=-f'(z)$, $\omega_y=-f'(x)$, $\omega_z=-f'(y)$. Since $f'(x)=(u_0/a)(1-2x^2/a^2)\exp(-x^2/a^2)$, it follows that the vorticity has two nulls at $x=y=z=\pm a/\sqrt{2}\equiv a_\pm$. It is easy to see that the null $x=y=z=a_+$ is of the type A_s, whereas the null $x=y=z=a_-$ is of the type B_s [11-13]. A null of type B_s is characterized by one real, negative eigenvalue and two complex eigenvalues. The two vortex lines originating from the B_s null along the eigenvector for the real, negative eigenvalue form the γ_{B_s} curve which is a stable manifold. The two-dimensional surface Σ_{B_s} which contains the eigenvectors for the two complex eigenvalues in the vicinity of the B_s null is an unstable manifold. As illustrated in Ref. [13], the vortex lines for a B_s null can be obtained simply by reversing the directions of the vortex lines for an A_s null. The straight line connecting the two vorticity nulls in our initial condition intersects the origin which is a null for the total velocity field.

The geometry of the flow fields we have described above is inspired by the pioneering work of Greene [12], developed further by Lau and Finn [13], on three-dimensional magnetic reconnection. Subsequently, Greene [14] has emphasized the role of nulls in three-dimensional vortex reconnection.

We now consider the time evolution of the initial state described above. It is intuitively clear that if the system is left to itself, it will tend to concentrate the vorticity near the velocity null. We seek solutions of the form $u_x(\mathbf{x},t)=u_2(y/a_2)\exp(-y^2/a_2^2)$, $u_y(\mathbf{x},t)=u_3(z/a_3)\exp(-z^2/a_3^2)$, and $u_z(\mathbf{x},t)=u_1(x/a_1)\exp(-x^2/a_1^2)$, where we require the functions u_1,u_2,u_3 and a_1,a_2,a_3 to change as functions of space and time in accordance with the incompressible Euler equations. These functions are constrained by means of an ordering procedure which exploits the separation in scales between the background and the localized flows. We take $a_1,a_2,a_3=O(a_0)$, where a_0 is a characteristic short scale, and continue to represent the long scale by L. Though the background flow is taken to be irrotational in the vicinity of the origin, it need not be irrotational far away from the origin. We shall call the region $|\mathbf{x}|\sim L\gg a_0$ the outer region. In the region $|\mathbf{x}|\lesssim a_0$, the localized flow obeys the equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{u}+\mathbf{v})\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla(\mathbf{u}+\mathbf{v}), \quad (1)$$

where $\boldsymbol{\omega}=\nabla\times\mathbf{u}$. The physical assumption underlying Eq. (1) is that the background flow $\mathbf{v}$ plays a passive role in the region $|\mathbf{x}|\lesssim a_0$, and that the crucial dynamics is controlled by the self-consistent evolution of $\mathbf{u}$ and $\boldsymbol{\omega}$. Also, there is no contribution from the large-scale vorticity Ω in Eq. (1) because the large-scale flow is irrotational near the origin. The region $|\mathbf{x}|\sim a_0$, which we call the middle region, is characterized by the relation $|\mathbf{u}|\sim|\mathbf{v}|$. In this region, we approximate the background flow $\mathbf{v}$ by the leading order term in a Taylor series which give $v_x\simeq(v_0/L)(y+z)$, $v_y\simeq(v_0/L)(z+x)$, and $v_z\simeq(v_0/L)\times(x+y)$. In the inner region $|\mathbf{x}|\ll a_0$ which includes the origin, Eq. (1) reduces to

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\mathbf{u}. \quad (2)$$

We formally introduce a small, positive dimensionless parameter $\varepsilon\equiv a_0/L$, order $|u|/|\mathbf{v}|\simeq1$, and carry out a multiple-scale analysis of the Euler equations. We define the (dimensionless) length variables $\mathbf{X}=(X,Y,Z)\equiv(x/L,y/L,z/L)$ and $\mathbf{X}_n=(X_n,Y_n,Z_n)\equiv(x/a_1,y/a_2,z/a_3)$. In the inner region $X\ll X_n$, $Y\ll Y_n$, and $Z\ll Z_n$. Since $|\mathbf{u}|\sim|\mathbf{v}|$ in the region $|\mathbf{x}|\sim a_0$, we define $\mathbf{X}_m=(X_m,Y_m,Z_m)$ by the relations

$$X_m\equiv(L/a_1)(u_1/v_0)(x/a_1)\exp(-x^2/a_1^2), \quad (3a)$$

$$Y_m\equiv(L/a_2)(u_2/v_0)(y/a_2)\exp(-y^2/a_2^2), \quad (3b)$$

$$Z_m\equiv(L/a_3)(u_3/v_0)(z/a_3)\exp(-z^2/a_3^2), \quad (3c)$$

such that, in the middle region, $|\mathbf{X}_n|\sim|\mathbf{X}_m|\sim1$. In this region, the background and localized flows can be written, respectively, as $\mathbf{v}\simeq\omega_0(a_2Y_n+a_3Z_n,a_3Z_n+a_1X_n,a_1X_n+a_2Y_n)$ and $\mathbf{u}\simeq\omega_0(a_2Y_m,a_3Z_m,a_1X_m)$, where $\omega_0\equiv v_0/L$.

We introduce multiple time scales $\tau=t$ and $T=\varepsilon t$, and seek solutions of the form $a_i=a_0(\tau)+\varepsilon\tilde{a}_i(\mathbf{x},\tau)$ and $u_i=U_0(T)+\varepsilon\tilde{U}_i(\varepsilon\mathbf{x},T)$. We then have $\partial/\partial t=\partial/\partial\tau+\varepsilon\partial/\partial T$. The operator $\partial/\partial x$ is given by

$$\frac{\partial}{\partial x}\simeq\frac{1}{a_1}\left[\frac{\partial}{\partial X_n}+\frac{X_m}{X_n}(1-2X_n^2)\frac{\partial}{\partial X_m}\right]. \quad (4)$$

Analogous expressions hold for $\partial/\partial y$ and $\partial/\partial z$. It follows that

$$\omega_x\simeq-\omega_0(1-2Z_n^2)\frac{Z_m}{Z_n}. \quad (5)$$

In the rest of the paper, we shall write out explicitly only the x component of the vector equations, since the symmetry of the initial conditions guarantees that the y and z components will follow analogously.

We consider the x component of Eq. (1) and evaluate the different terms in it. It follows from Eq. (5) that $\partial\omega_x/\partial x=\partial\omega_x/\partial y=O(\varepsilon)$ and to $O(1)$,

$$\frac{\partial\omega_x}{\partial z}\simeq\frac{2\omega_0}{a_3}Z_m(3-2Z_n^2). \quad (6)$$

Similarly, we find that $\partial u_x/\partial x=\partial u_x/\partial z=O(\varepsilon)$ and to $O(1)$,

$$\frac{\partial u_x}{\partial y}\simeq\omega_0(1-2Y_n^2)\frac{Y_m}{Y_n}. \quad (7)$$

Using Eqs. (5)-(7), the x component of Eq. (1) yields to $O(1)$,

$$\frac{\partial a_3}{\partial\tau}\simeq-\omega_0[(1-2Z_n^2)^2-4Z_n^2]^{-1}\left[2a_1X_mZ_n(3-2Z_n^2)+2Z_n(a_1X_n+a_2Y_n)(3-2Z_n^2)+a_3\left\{(1-2X_n^2)(1-2Y_n^2)\frac{X_mY_mZ_n}{X_nY_nZ_m}+(1-2X_n^2)\frac{Z_nX_m}{Z_mX_n}+(1-2Y_n^2)\frac{Y_mZ_n}{Y_nZ_m}\right\}\right]. \quad (8)$$

Equation (8) describes the time evolution of the function a_3 in the middle region. Analogous equations may be obtained for the functions a_1 and a_2 from the y and z components, respectively, of Eq. (1).

We now consider the inner region. For compactness of notation, we define $H(x)\equiv x\exp(-x^2)$. Since, in the inner region, we have $|x|\ll a_0$, it follows that $|X_n|\to0$, $H(X_n),H''(X_n)\to0$ and $H'(X_n)\to1$. From the x component of Eq. (3),

$$\frac{\partial a_3}{\partial \tau} \simeq [Z_n H''(Z_n) + H'(Z_n)]^{-1} \left[u_1 H(X_n) H''(Z_n) - \frac{a_3^2}{a_1 a_2} \frac{u_1 u_2}{u_3} H'(X_n) H'(Y_n) \right], \tag{9}$$

we obtain, in the limit $X_n \to 0$,

$$\frac{\partial a_3}{\partial \tau} \simeq -\frac{a_3^2}{a_1 a_2} \frac{u_1 u_2}{u_3}. \tag{10}$$

We now match asymptotically the solutions in the inner and middle regions. The matching condition is

$$\lim_{|\mathbf{x}| \to 0} \left. \frac{\partial a_3}{\partial \tau} \right|_{\text{middle}} = \lim_{|\mathbf{x}| \to \infty} \left. \frac{\partial a_3}{\partial \tau} \right|_{\text{inner}}. \tag{11}$$

In order to implement the matching condition (11), we note that $(X_m, Y_m, Z_m) \to (U_0/a_0\omega_0)(X_n, Y_n, Z_n) \gg (X_n, Y_n, Z_n)$ because $(U_0/a_0\omega_0) \sim (|\mathbf{u}|/|\mathbf{v}|)L/a_0 \sim 1/\varepsilon$. Hence, setting $u_1, u_2, u_3 \to U_0$, $a_1, a_2, a_3 \to a_0$, and $(X_m, Y_m, Z_m) \to (U_0/a_0\omega_0)(X_n, Y_n, Z_n)$, we get to leading order

$$\left. \frac{\partial a_0}{\partial \tau} \right|_{\text{inner}} \simeq -\frac{a_0 Z_n}{Z_m} \omega_0 \frac{X_m Y_m}{X_n Y_n} \simeq -U_0. \tag{12}$$

Integrating Eq. (12) with respect to the "fast" time $\tau = t$, we get $a_0 \simeq U_0(t_c - t)$, where $t_c \simeq a_0(t=0)/U_0$. Hence, $\omega_x \sim (t_c - t)^{-1}$, which gives a finite-time singularity at $t = t_c$. It follows that $\omega_y \sim \omega_z \sim (t_c - t)^{-1}$.

In order to complete the solution, we must match asymptotically the solutions in the middle and outer regions. The leading order equation in the outer region is $\partial \mathbf{\Omega}/\partial t + \mathbf{v} \cdot \nabla \mathbf{\Omega} = \mathbf{\Omega} \cdot \nabla \mathbf{v}$. Clearly, neither $\mathbf{u}$ nor $\boldsymbol{\omega}$ enters at this order. To find $\boldsymbol{\omega}$ in the outer region, we must go beyond this "negligibly large" equation to the "small" equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{\Omega} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = \mathbf{\Omega} \cdot \nabla \mathbf{u} + \boldsymbol{\omega} \cdot \nabla \mathbf{v}. \tag{13}$$

We consider now the x component of Eq. (13). For $\varepsilon \ll 1$, it is easy to see that $|\mathbf{v} \cdot \nabla \omega_x| \gg |\mathbf{\Omega} \cdot \nabla v_x| \sim |\boldsymbol{\omega} \cdot \nabla u_x| \gg |\mathbf{u} \cdot \nabla \Omega_x|$. Hence, the x component of Eq. (13) reduces to

$$\frac{\partial \omega_x}{\partial t} + \mathbf{v} \cdot \nabla \omega_x \simeq 0. \tag{14}$$

[The y and z components of Eq. (13) are, respectively, the y and z components of $\partial \boldsymbol{\omega}/\partial t + \mathbf{v} \cdot \nabla \boldsymbol{\omega} \simeq 0$.] Equation (14) gives

$$\frac{\partial a_3}{\partial \tau} \simeq \frac{a_3}{z} F(x+y). \tag{15}$$

The asymptotic matching condition between the middle and outer regions is

$$\lim_{|\mathbf{x}| \to \infty} \left. \frac{\partial a_3}{\partial \tau} \right|_{\text{middle}} = \lim_{|\mathbf{x}| \to 0} \left. \frac{\partial a_3}{\partial \tau} \right|_{\text{outer}}. \tag{16}$$

We now check that Eqs. (8) and (15) do indeed satisfy the matching condition (16). We recall that $F(x+y) \simeq \omega_0(x+y)$ as $|\mathbf{x}| \to 0$. This reduces the right-hand side of Eq. (16) to $\omega_0 a_3(x+y)/z$. Now, using Eq. (8), we see that as $|\mathbf{X}_n| \to \infty$, $|\mathbf{X}_m| \to 0$, the left-hand side of Eq. (16) reduces to $\omega_0(a_1 X_n + a_2 Y_n)/Z_n = a_3(x+y)/z$. Hence, the matching condition (16).

The symmetry of the initial conditions singles out the $x = y = z$ line as a natural axis. It is instructive to review the inner region solution in a new coordinate system (x', y', z') where $\hat{\mathbf{z}}' = (1/\sqrt{3})(1,1,1)$ and $\hat{\mathbf{x}}', \hat{\mathbf{y}}'$ are two mutually orthogonal unit vectors in the plane normal to $\hat{\mathbf{z}}'$. To be specific, we take $\hat{\mathbf{x}}' = (1/\sqrt{6})(-1,-1,2)$ and $\hat{\mathbf{y}}' = (1/\sqrt{2})(1,-1,0)$. To leading order, we get

$$\mathbf{u} = \begin{pmatrix} u_{x'} \\ u_{y'} \\ u_{z'} \end{pmatrix} = \frac{U_0}{a_0} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \cdots. \tag{17}$$

Redefining $x' = r'\cos\theta'$, $y' = r'\sin\theta'$, Eq. (17) becomes

$$\mathbf{u} = \begin{pmatrix} u_{r'} \\ u_{\theta'} \\ u_{z'} \end{pmatrix} = \frac{U_0}{a_0} \begin{pmatrix} -\frac{r'}{2} \\ \frac{\sqrt{3}}{2} r' \\ z' \end{pmatrix} + \cdots \tag{18}$$

which is axisymmetric to leading order. The departure from axisymmetry and the spiral structure is manifest at higher order. If Eq. (17) is carried through to higher order, we get

$$\boldsymbol{\omega} = \sqrt{3} \frac{U_0}{a_0} \left[-\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{z'}{a_0^2} \begin{pmatrix} -r' \\ \sqrt{3} r' \\ -z' \end{pmatrix} + \frac{r'^2}{a_0^2} \begin{pmatrix} \frac{\sqrt{3}}{2} \sin\left(3\theta' - \frac{\pi}{6}\right) \\ \frac{\sqrt{3}}{2} \cos\left(3\theta' - \frac{\pi}{6}\right) \\ -1 \end{pmatrix} \right] + \cdots. \tag{19}$$

In the new coordinate system, the inner region equation (2) reduces, in leading order, to

$$\frac{\partial \omega_{z'}}{\partial t} \simeq \omega_{z'} \frac{\partial}{\partial z'} u_{z'} \tag{20}$$

which, in turn, gives $\partial a_0/\partial t \simeq -U_0$ and hence, $a_0 \simeq U_0(t_c - t)$, as obtained earlier.

In a small region surrounding the origin, the solutions of the Euler equations exhibit a local self-similarity. To be precise, for all times prior to the blowup of vorticity, there exists a small region surrounding the origin where the solution is invariant under the scaling $\mathbf{x} \rightarrow c\mathbf{x}$, $\mathbf{u} \rightarrow \mathbf{u}$, $t \rightarrow ct$, where c is a constant. In this small region, the solutions $u_x = y/(t_c - t)$, $u_y = z/(t_c - t)$, $u_z = x/(t_c - t)$ [which imply $\omega_x = \omega_y = \omega_z = -1/(t_c - t)$] satisfy Eq. (2) exactly.

It may be argued, on first glance, that the singularity obtained here is unphysical because the initial flows have infinite total energy. This is indeed the criticism [15] of earlier investigations of two-dimensional [15] and three-dimensional [16] solutions of the stagnation-point form. We point, however, to an important difference between our system of flows and those considered in Refs. [15] and [16]: In our initial conditions, the velocity and vorticity fields are bounded everywhere, including points at infinity. Infinite energy is obtained in our initial conditions merely because our system size is infinite. We emphasize that the velocity itself remains bounded in our model for all times leading to the blowup of the vorticity. Since the energy density is finite everywhere, including points at infinity, we can exclude the possibility that the finite-time singularity is an artifact of the infinite system size [17].

Looking back at our derivation of the vortex singularity, one may wonder as to why we retain the background flow $\mathbf{v}$ at all, since it plays only a passive, uninteresting role. We do so because it sets the large scale L, and also because it is likely to be present in most physical situations of interest. The background flow $\mathbf{v}$ would play a more active role if, in our initial state, $|\mathbf{u}|$ were much smaller than $|\mathbf{v}|$. Then it can be shown that the background flow would shrink a_0 (exponentially under certain conditions) and enhance $\mathbf{u}$ until the condition $|\mathbf{u}| \sim |\mathbf{v}|$ is realized. Once this occurs, the background flow would again be reduced to a passive role, as in our present calculation.

The finite-time singularity obtained in our model should be observable in numerical experiments involving antiparallel [5,10,18–20] or orthogonal [21,22] vortex tubes. In such interactions, the configuration of a velocity null on a null-null line for the vorticity can occur naturally. The singularity is then realized as the two vorticity nulls approach the velocity null.

The singularity obtained in the present model is fully three dimensional in character. Yet the occurrence of spiral structures is reminiscent of the strained vortex model of Lundgren [23] which, in turn, can be viewed as a natural sequel to Townsend's model [24] of the intermittent structure of turbulence as being due to a random distribution of vortex tubes (and sheets) each of which is subjected to a background strain caused by other vortex structures. The presence of even a small but finite viscosity is expected to arrest the formation of the finite-time singularity in our model.

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