Arithmetical Chaos and Violation of Universality in Energy Level Statistics

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A class of strongly chaotic systems revealing a strange arithmetical structure is discussed whose quantal energy levels exhibit level attraction rather than repulsion. As an example, the nearest-neighbor level spacings for Artin's billiard have been computed in a large energy range. It is shown that the observed violation of universality has its root in the existence of an infinite number of Hermitian operators (Hecke operators) which commute with the Hamiltonian and generate nongeneric correlations in the eigenfunctions.

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There seems to be no doubt that the statistical properties of quantal energy spectra and eigenvectors of classically chaotic systems are well described (with respect to their short-range correlations) by the universal laws of random-matrix theory (RMT) [1], originally proposed by Wigner and by Landau and Smorodinsky and fully developed by Dyson for a better understanding of the resonances of compound nuclei. (See Ref. [2] for a collection of the original papers, and Refs. [1,3,4] for recent reviews.) In the simplest case of generic Hamiltonian systems whose classical motion is time-reversal invariant, two *universality classes* are predicted: All systems with an integrable classical limit fall into one class, and all systems whose classical limit is strongly chaotic (K systems) fall into another class. Consider, e.g., the density distribution $P(s)$ of "unfolded" nearest-neighbor level spacings s. In accordance with the empirical observation that the short-range spectral fluctuations of integrable systems are just those of random numbers, the level spacing distributions for these systems are Poissonian, i.e., $P_{\rm Poisson}(s)=e^{-s}$, while those of chaotic systems are the same as for the eigenvalues of large real symmetric random matrices, i.e., of the Gaussian orthogonal ensemble (GOE), which in the case of $P(s)$ is well approximated by Wigner's surmise $P_{\rm Wigner}(s)=\frac{1}{2}\pi s e^{-\pi s^2/4}$. For small spacings, $s\rightarrow 0$, the two distributions exhibit the well-known phenomenon of *level attraction* $[P(s)\sim 1-s]$ for classically integrable systems and of *level repulsion* $[P(s)\sim \frac{1}{2}s]$ for strongly chaotic systems. It thus appears that the statistical properties of a given quantum system are already determined by its classical limit, depending only upon whether this is chaotic or not.

In this Letter we study an interesting class of strongly chaotic systems whose quantal eigenvalues exhibit surprisingly enough level attraction rather than repulsion, and which thus lead to an apparent violation of the universal laws of RMT. We show that these systems reveal a strange arithmetical structure of chaos, which we call *arithmetical chaos*, that manifests itself in the existence of an infinite number of Hermitian operators commuting with the quantum Hamiltonian, and which are the origin of unexpected correlations in the quantum eigenstates. The existence of such operators could not be anticipated since these systems, having 2 degrees of freedom and being ergodic, possess classically only a single constant of motion, namely, the conserved energy. Before giving a general characterization of such systems, we discuss in some detail Artin's billiard as a prototype example.

Artin's billiard [5,6] is a two-dimensional non-Euclidean billiard consisting of a point particle sliding freely on a noncompact Riemannian surface of constant negative Gaussian curvature $K=-1$ with the topology of a sphere containing an open end (cusp) at infinity. The surface can be realized on the Poincaré upper half plane $\mathcal{H}=\{z=x+iy,\ y>0\}$ endowed with the hyperbolic metric $ds^2=y^{-2}(dx^2+dy^2)$. On $\mathcal{H}$ the modular group $\Gamma={\rm PSL}(2,\mathbb{Z})$ operates via fractional linear transformations, i.e., by $\gamma=(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})\in\Gamma$, $z\in\mathcal{H}$, $\gamma z=(az+b)/(cz+d)$. The classical (i.e., geodesic) motion on the surface $\Gamma\backslash\mathcal{H}$ is rigorously known to be ergodic [5,7] and even strongly chaotic (K system). This surface can also be represented by the fundamental region $\mathcal{F}=\{|z|\geq 1,\ -\frac{1}{2}\leq x\leq\frac{1}{2}\}\subset\mathcal{H}$ of the modular group, provided the boundaries are identified correctly. $\mathcal{F}$ is a noncompact triangle of finite area $\pi/3$. The corresponding quantum system is governed by the Schrödinger equation $H\psi=E\psi$ with the Hamiltonian $H=-\Delta$, where $\Delta=y^2(\partial_x^2+\partial_y^2)$ is the hyperbolic Laplacian ($\hbar=1=2m$). The eigenfunctions have to be invariant under the action of the modular group, i.e., they have to obey the periodic boundary conditions (b.c.) $\psi(\gamma z)=\psi(z)$ for all $\gamma\in\Gamma$ and $z\in\mathcal{H}$. The spectrum of H is both continuous and discrete. (See Ref. [8] for the mathematical background.) The scattering solutions corresponding to the continuous spectrum are explicitly known and are given by the Einstein series. Here we are interested only in the eigenfunctions $\psi_n(z)$ which are square integrable with respect to the scalar product $\langle\psi,\phi\rangle=\int_{\mathcal{F}}\bar{\psi}\phi\,dx\,dy/y^2$ and thus belong to the discrete spectrum of H. Artin's billiard is symmetric under reflections on the imaginary axis. We therefore consider the two *desymmetrized* quantum billiards corresponding to the two *parity classes* $\psi_n^\pm(-\bar z)=\pm\psi_n^\pm(z)$, depending on whether the eigenfunctions are even or odd under reflections on the line $x=0$. The desymmetrized systems can be viewed as quantum billiards defined on the halved domain $\mathcal{D}=\{|z|\geq 1,\ 0\leq x\leq \frac{1}{2}\}$. Then the eigenfunctions have to obey Neumann b.c. on $\partial\mathcal{D}$ for parity $+1$ and Dirichlet b.c. on $\partial\mathcal{D}$ for parity -1. Apart from the ground state with energy $E_0^+=0$ belonging to the constant eigenfunction $\psi_0^+=(3/\pi)^{1/2}$, the discrete spectrum $E_1^\pm\leq E_2^\pm\leq\cdots$ is embedded in the continuum $[\frac{1}{4},\infty)$ as can be seen from the rigorous lower bound $E_1^\pm>3\pi^2/2>\frac{1}{4}$. Expressing the eigenvalues E_n by the momenta p_n, $E_n=p_n^2+\frac{1}{4}$, $n\geq 1$, the even eigenfunctions have the Fourier expansion (suppressing the label $+$)

$$\psi_n(z)=N_n\sqrt{y}\sum_{k=1}^{\infty}c_n(k)K_{ip_n}(2\pi ky)\cos(2\pi kx)\,, \qquad (1)$$

while the odd eigenfunctions have a similar expansion with $\sin(2\pi kx)$ replacing $\cos(2\pi kx)$. From the theory of *Hecke operators* (see below) it follows that the expansion coefficients $c_n(k)$ are real with $c_n(1)\neq 0$ for all $n\geq 1$, which allows us to work with *normalized* coefficients satisfying $c_n(1)=1$. The real overall normalization N_n in Eq. (1) ensures that $\langle\psi_n,\psi_m\rangle=\delta_{nm}$. Notice that the ψ_n's are real due to a well-known property of the modified Bessel function $K_\nu(x)$. In the mathematical literature the eigenfunctions (1) are known as nonholomorphic cusp forms or Maass wave forms. The coefficients $c_n(k)$ satisfy the exact upper bound $|c_n(k)|\leq d(k)k^{1/4}$, where $d(k)$ denotes the number of positive divisors of k. With $d(k)\leq 2\sqrt{k}$ and $K_{ip_n}(2\pi ky)=O(e^{-2\pi ky})$ it follows that the series expansion (1) is absolutely convergent in the whole billiard domain. The generalized Ramanujan-Petersson conjecture asserts that $|c_n(k)|\leq d(k)$, i.e., in particular, $|c_n(p)|\leq 2$ for every prime p.

At present, no analytical results are known for the eigenvalues E_n or the coefficients $c_n(k)$ for Artin's billiard despite the great importance such results could have for modern number theory [8,9]. Recently it was demonstrated [6,10] that the low-lying energy levels can be determined approximately by computing the nontrivial zeros of a certain Selberg zeta function or by using a special version of the Selberg trace formula, respectively. (Similar results hold for other chaotic systems, too, but then, in general, only in the semiclassical limit.) Until fairly recently, there were major difficulties associated with computing the eigenvalues by solving the Schrödinger equation. (See Appendix C of Hejhal's treatise [8] for various papers having appeared since the early 1970's.) A major breakthrough has been achieved by Hejhal [9] (see, also, Ref. [11]) who was able to compute with high precision all eigenvalues with momenta $p_n\leq 50.1$ corresponding to 50 even and 73 odd eigenvalues. The nearest-neighbor level spacing distribution $P(s)$ computed from this small sample of eigenvalues did not show the expected level repulsion and thus provided the first hint [12] that Artin's billiard does not fit into the universal scheme of RMT. To obtain clear evidence for the violation of universality one requires a larger number of energy levels. Recently, we have determined [13] the eigenvalues E_n with an accuracy of about 10^{-10} together with the first coefficients $c_n(k)$ for momenta $p_n\leq 300$ corresponding to the first 3167 even and 3475 odd eigenfunctions. No degeneracy occurs and no violation of the Ramanujan-Petersson conjecture is observed. (Independent computations have been carried out for a few low-lying levels also in [14] and for the first odd levels in [15].) With the help of the improved Weyl's law counting the energy levels E_n [see Eq. (9) in Ref. [6] for the odd and [13,16] for the even case] we have unfolded the two spectra and computed the level spacing distributions $P(s)$. In Fig. 1 we present the histograms computed from the eigenvalues in the momentum range $250\leq p_n\leq 300$; Fig. 1(a) corresponds to the even case comprising 1026 levels, while Fig. 1(b) refers to the 1093 odd levels. The

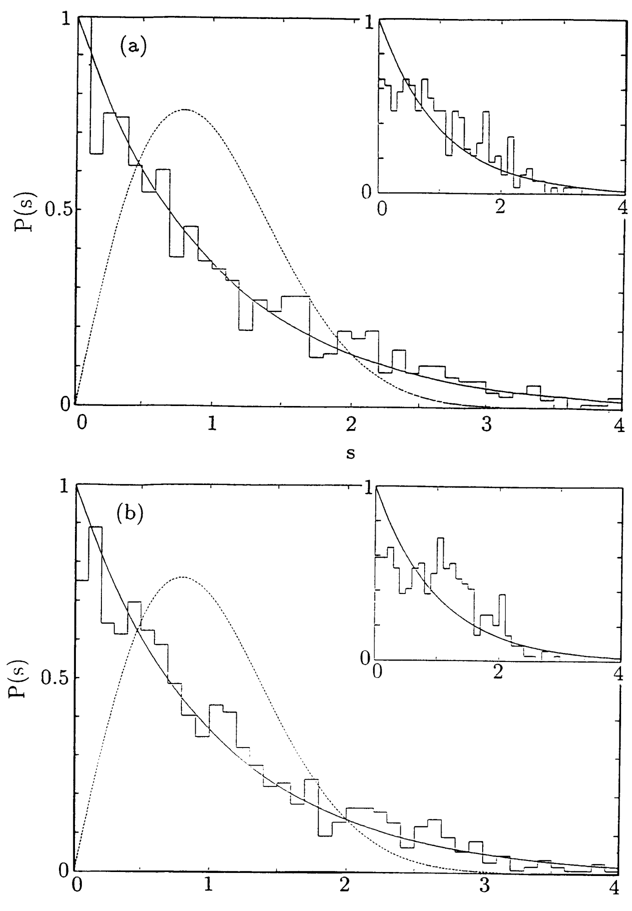

FIG. 1. Histograms of the level spacing distributions $P(s)$ for Artin's billiard for (a) even and (b) odd eigenfunctions in the momentum range $250\leq p_n\leq 300$. The solid and dashed lines correspond to the Poisson and GOE predictions, respectively. Insets: $P(s)$ in the low momentum range $p_n\leq 100$.

two universal statistics are plotted as the solid and dashed lines, respectively. One observes a striking level attraction and thus a clear violation of the universal laws of RMT. We have also calculated $P(s)$ for various samples of levels at lower energies; again the Wigner distribution is ruled out, although at small spacings the histograms are somewhat lower, indicating that the distributions are not yet stationary. The two insets in Fig. 1 show the histograms corresponding to the low momentum range $p_n \leq 100$. Summarizing our findings we can safely say that Artin's billiard, although strongly chaotic, reveals at high energies Poissonian fluctuations, even after complete desymmetrization. We will now show that this strange result has its root in a deep result of number theory associated with the theory of Hecke operators.

Define for any $k \in \mathbb{N}$ the *Hecke operator* T_k acting on a Maass wave form $\phi(z)$ by

$$T_k\phi(z) = \frac{1}{\sqrt{k}} \sum_{\substack{ad=k,d>0 \\ b \bmod d}} \phi\left(\frac{az+b}{d}\right). \tag{2}$$

Then it can be shown [8] that the algebra generated by the T_k's is commutative, $[T_k,T_l]=0$, with $T_kT_l = \sum_{d|(k,l)} T_{kl/d^2}$, where (k,l) denotes the largest common divisor of k and l. Furthermore, acting on Maass wave forms, the Hecke operators are Hermitian, $\langle\psi,T_k\phi\rangle = \langle T_k\psi,\phi\rangle$, and since $[\Delta,T_k]=0$ holds, one infers that the eigenfunctions (1) of our Hamiltonian can be chosen to be simultaneously eigenfunctions of all Hecke operators leading to the remarkable relation $T_k\psi_n(z) = c_n(k)\psi_n(z)$ for all $k,n \in \mathbb{N}$, which tells us that the normalized coefficients $c_n(k)$ are just the (real) eigenvalues of the Hecke operators. Most important for our discussion are the *Hecke relations*

$$c_n(k)c_n(l) = \sum_{d|(k,l)} c_n\left(\frac{kl}{d^2}\right), \tag{3}$$

which are a direct consequence of the multiplication law for the T_k's, and which state that the expansion coefficients of the eigenstates (1) are highly *correlated.* In fact, the $c_n(k)$ are polynomials in $c_n(p)$, p prime. [E.g., $c_n(4) = c_n^2(2) - 1$, $c_n(6) = c_n(2)c_n(3)$, $c_n(8) = c_n^3(2) - 2c_n(2)$, $c_n(12) = c_n(3)c_n(4)$.] This is in striking contrast to the prediction of RMT that the expansion coefficients with respect to a generic basis should be Gaussian distributed with a zero mean. Although we have chosen a special basis in (1), the correlations will pertain in a generic basis since the Hecke operators relate the values of the wave functions at different points [see Eq. (2)]. Thus we have shown that the eigenfunctions of Artin's billiard do not fit into the universal scheme of RMT. In view of this result there is no reason why the level fluctuations should obey the GOE prediction and, indeed, they do not, as demonstrated in Fig. 1.

Finally, we would like to point out that Artin's billiard is not a singular exception but rather belongs to a large family of strongly chaotic systems sharing similar properties. Consider the geodesic motion on compact or noncompact Riemannian surfaces M with constant negative curvature and of finite area. Such surfaces can be represented as $M = \Gamma\backslash\mathcal{H}$, where Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Some of the simplest generalizations of Artin's billiard are obtained by considering [11,14] as "billiard tables" the noncompact domains $\mathcal{F}_N = \{|z| \geq 1, |x| \leq \cos(\pi/N)\}$, $N = 3,4,6,\infty$, which are the fundamental regions of Hecke triangle groups. (The case $N=3$ is just Artin's billiard.) For definiteness, let us now concentrate on the motion on compact surfaces of genus $g \geq 2$ (Hadamard-Gutzwiller model [17]) realized on $\mathcal{H}$ by hyperbolic polygons with $4g$ edges. Again, the Hamiltonian is given by $H = -\Delta$, and the eigenfunctions have to satisfy periodic b.c., i.e., $\psi(\gamma z) = \psi(z)$ for all $\gamma \in \Gamma$ (Γ-automorphic functions). Then H has only a discrete spectrum $0 = E_0 < E_1 \leq E_2 \leq \dots$. Among the infinitely many inequivalent choices for Γ, the so-called *arithmetical* Fuchsian groups [18] play a special role. These groups have the special property that all traces of the 2×2 matrices representing a given $\gamma \in \Gamma$ are algebraic integers, and thus the lengths l_n of the classical periodic orbits (closed geodesics) are determined by algebraic integers, too, in the form $2\cosh(l_n/2) =$ *algebraic integer.* This leads to a strange *arithmetical structure of chaos,* first discovered in [19] in the case of the "regular octagon" corresponding to the simplest case $g=2$ and associated with the highest possible symmetry. These arithmetical properties made it possible to construct an explicit enumeration scheme for the group elements and thus to calculate completely the shortest ca. 4×10^6 periodic orbits [20]. Surprisingly, an exponential increase of the mean multiplicity $\bar{g}_n$ of the lengths of primitive periodic orbits was found, $\bar{g}_n \sim 8\sqrt{2}e^{l_n/2}/l_n$. We now recognize that $\bar{g}_n \sim \text{const} \times e^{l_n/2}/l_n$ is a universal feature of arithmetical chaos due to the above-mentioned arithmetical structure of the length spectrum. (Notice that $g_n = 1$ or 2 for generic systems with time-reversal invariance.) Quite analogously to what has been observed in Artin's billiard, the energy levels of the regular octagon show level attraction rather than repulsion, even after complete desymmetrization [21]. (For a special symmetry class, this was first observed by Bohigas, Giannoni, and Schmit [22].) It has been argued (see the contributions by Bohigas and by Schmit in [4]) that the violation of the universal laws of RMT is caused by the fact that the regular octagon is tessellating the hyperbolic plane and thus Gutzwiller's trace formula [3] is exact rather than a semiclassical approximation as in the generic case, since it is identical to the Selberg trace formula. This explanation is not correct, however, since it has been shown [23] that the short-range fluctuations of thirty different Hadamard-Gutzwiller models on asymmetric surfaces $\Gamma\backslash\mathcal{H}$ generated *by nonarithmetical groups* Γ are in nice agreement with the GOE predictions. For all these systems the Selberg trace formula holds, and all these octagons tessellate the hyperbolic plane. Thus there must be a different explanation. Indeed, all arithmetical Fuchsian groups possess [18] an infinite number of Hecke operators that are Hermitian and commute with the Hamiltonian and which generate correlations in the eigenfunctions. Therefore, the situation is analogous to Artin's billiard, and no GOE fluctuations are expected.

In this Letter we have discussed a class of strongly chaotic systems whose classical motion exhibits arithmetical chaos and whose quantal energy levels show level attraction rather than repulsion. It has been pointed out that this striking violation of the universal laws of RMT has its root in the existence of an infinite number of Hecke operators which generate nongeneric correlations in the quantum wave functions.

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