Universal Criterion for the Breakup of Invariant Tori in Dissipative Systems

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The transition from quasiperiodicity to chaos is studied in a two-dimensional dissipative map with the inverse-golden-mean rotation number. On the basis of a decimation scheme, it is argued that the (minimal) slope of the critical iterated circle map is proportional to the effective Jacobian determinant. Approaching the zero-Jacobian-determinant limit, the factor of proportion becomes a universal constant. Numerical investigation on the dissipative standard map suggests that this universal number could become observable in experiments.

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In an almost linear continuous dynamical system with two competing frequencies, the attractor is typically either a mode-locked periodic state or an invariant torus associated with quasiperiodic motion. In the parameter space, periodic and quasiperiodic attractors are mingled up in such a way that there is a positive probability for each kind of orbit to be found. Experiments indicate that mode locking and quasiperiodic behavior are generic in hydrodynamics [1], charge-density-wave conductors [2], and other physical [3,4] and chemical [5] systems. Changing parameter values may increase the nonlinearity and lead to a transition to chaos.

The experiment by Martin and Martienssen [3] on the electrical conductivity of barium sodium niobate crystals is a very nice example of the case in which it is possible to measure the actual return map characterizing the discretized dynamics on an invariant circle (the invariant torus appears as an invariant circle for the Poincaré map of the system). Bohr *et al.* [6,7] point out the intimate connection between the existence of an invariant circle and the one-dimensional nature of the return map. In particular, they show that a zero slope in the return map is impossible if the underlying invariant circle is smooth. The fact that the invariant circle loses smoothness before breaking up [8,9] could mislead one into thinking that a zero slope in the return map is a necessary condition for the system to be critical, i.e., about to become chaotic. Another motivation for this kind of false idea could come from the fact that an analytic circle map has a zero-slope inflection point at the transition from quasiperiodicity to chaos [10]. However, if the Jacobian determinant of the Poincaré map is positive everywhere along the invariant circle, it is impossible that the "reduced" circle map, i.e., the projection of the Poincaré map on the invariant circle, would have a zero slope at some point of the circle. The reduced circle map could have a zero slope only if the tangent vector at the corresponding point was annihilated by the Jacobian matrix. This could happen only if the Jacobian determinant vanished. The positivity of the slope of a critical circle map has been noticed by several authors [11].

In this Letter the relation between the slope of the reduced circle map and the Jacobian determinant is elaborated further. For simplicity, I will restrict myself to a two-dimensional Poincaré map with rotation number $\zeta=(\sqrt{5}-1)/2$. The reduced circle map is denoted by $h(x)$, where x is a scaled "angle" variable for the invariant circle so that $h(x+1)=h(x)+1$. By the assumption on the rotation number, $h^n(x)/n \to \xi$ and, moreover, $h_n(x)-F_{n-1} \equiv h^{F_n}(x)-F_{n-1} \to x$ as n tends to infinity. Here F_n stands for the nth Fibonacci number, $F_{n+1}=F_n+F_{n-1}$ ($F_0=0$, $F_1=1$). The Jacobian determinant of the F_n-times iterated map at x is denoted by $J_n(x)$. It will be shown below that in the critical case $h_n'(x_0) \sim J_n(x_0)$, where x_0 is a special point [8,9] associated with the universal scaling by $\alpha \approx -1.2885746$. x_0 corresponds to a cubic critical point for an analytic circle map. In a higher-dimensional dissipative system, x_0 can be searched either as the point visited most rarely by the quasiperiodic orbit or as the limit of points x_n, $n \to \infty$, such that $h_n'(x)$ has a minimum at x_n. It could as well be stated that $h_n'(x_n) \sim J_n(x_n)$. For a dissipative system, $J_n(x) \to 0$ as $n \to \infty$, so that a zero slope is indeed observed but only considering the limit of an infinitely high iterate of the original circle map.

Furthermore, the calculation shows that the factor of proportion between $h_n'(x_0)$ and $J_n(x_0)$ tends to a universal constant as the Jacobian determinant approaches zero. In this limit,

$$\frac{h_n'(x_0)}{J_n(x_0)} \to a = \frac{2}{[\eta(0)]^2 \eta'''(0)} \quad \text{as } n \to \infty , \tag{1}$$

where $\eta(x)$ is one of the components of the universal fixed-point pair (ξ,η) for the standard renormalization operator $T(\xi,\eta)=\alpha(\eta,\eta\circ\xi)\alpha^{-1}$ for analytic circle maps [8]. MacKay's [12] expansion for $\eta(x)$ leads to the numerical estimate $a \approx 0.435625$.

The starting point is Bohr's [7] formula relating the derivatives of the first and the second iterates of the reduced circle map. Consider a two-dimensional map $G(x,y)=(g_1(x,y),g_2(x,y))$ with an invariant circle $y=c(x)$ $[G(x+1,y)=G(x,y)+(1,0)]$. The original map can be related to a one-dimensional circle map by $h(x)=g_1(x,c(x))$ and $c(h(x))=g_2(x,c(x))$. Differentiating these two equations with respect to x yields, after some manipulation,

$$\frac{dh^2(x)}{dx}=\left[g_{11}(h(x))+\frac{g_{22}(x)g_{12}(h(x))}{g_{12}(x)}\right]\times h'(x)-\frac{g_{12}(h(x))}{g_{12}(x)}J(x)\,, \tag{2}$$

where

$$g_{i1}(x)=\left.\frac{\partial g_i(x,y)}{\partial x}\right|_{y=c(x)},\quad g_{i2}(x)=\left.\frac{\partial g_i(x,y)}{\partial y}\right|_{y=c(x)},$$

$$J(x)=g_{11}(x)g_{22}(x)-g_{12}(x)g_{21}(x)\,.$$

It is important that the equation for the invariant circle does not appear in (2).

Equation (2) can be written in the form

$$h_3'(x)=c_2(x)h_2'(x)+d_2(x)\,. \tag{3}$$

I introduce a decimation technique by which one can generate from (3) and the trivial equation $h_2'(x)=1h_1'(x)+0$ a sequence of equations

$$h_{n+1}'(x)=c_n(x)h_n'(x)+d_n(x) \tag{4}$$

with increasing n. Assume that (4) is known for $n=i$ and $n=i-1$. Write $h_{i+2}'(x)$ as

$$h_{i+1}'(h_i(x))h_i'(x)=c_i(h_i(x))h_i'(h_i(x))h_i'(x)+d_i(h_i(x))h_i'(x) \tag{5}$$

and split $h_i'(h_i(x))$ further:

$$h_i'(h_i(x))=c_{i-1}(h_i(x))h_{i-1}'(h_i(x))+d_{i-1}(h_i(x))\,. \tag{6}$$

Replacing $h_{i-1}'(h_i(x))h_i'(x)$ by $h_{i+1}'(x)$ and using the fact that

$$h_i'(x)=\frac{h_{i+1}'(x)-d_i(x)}{c_i(x)}$$

leads finally to

$$h_{i+2}'(x)=c_{i+1}(x)h_{i+1}'(x)+d_{i+1}(x)$$

with

$$c_{i+1}(x)=c_i(h_i(x))c_{i-1}(h_i(x))-\frac{d_{i+1}(x)}{d_i(x)}\,,$$
$$d_{i+1}(x)=-\frac{d_i(x)[d_i(h_i(x))+c_i(h_i(x))d_{i-1}(h_i(x))]}{c_i(x)}\,. \tag{7}$$

These recursion relations help in determining the leading asymptotic behavior of $c_n(x)$ and $d_n(x)$ as n tends to infinity. First, it can be inductively argued that $d_n(x)\sim J_n(x)$ (except for $n=1$). Recall that $d_1(x)\equiv 0$ and $d_2(x)\sim J(x)$ so that (7) implies $d_3(x)\sim J(x)\times J(h(x))=J_3(x)$. At each level $n\geq 3$ of the recursion, the leading term in the Jacobian determinant arises from the product $\sim d_n(x)d_{n-1}(h_n(x))\sim J_n(x)J_{n-1}(h_n(x))=J_{n+1}(x)$. One can now proceed to determine the asymptotic behavior of $c_n(x)$ as $n\to\infty$. Equations (5)-(7) give

$$h_{n+1}'(h_n(x))=c_{n+1}(x)h_{n-1}'(h_n(x))+\frac{d_{n+1}(x)}{d_n(x)}[h_{n-1}'(h_n(x))-c_n(x)]\,.$$

Because $d_{n+1}(x)/d_n(x)\to 0$, all one needs to know is the asymptotic behavior of $h_{n+1}'(h_n(x))$ and $h_{n-1}'(h_n(x))$. I consider here only the critical case with $x=x_0$. On the basis of the renormalization theory [8,9,13], it is expected that

$$h_n(x_0+z)-F_{n-1}-x_0\approx\alpha^{-n}\eta(\alpha^n z)\,, \tag{8}$$

where the approximation improves with increasing n. This implies $h_{n+1}'(h_n(x_0))\to\eta'(\xi(0))=\alpha^4$ and $h_{n-1}'\times(h_n(x_0))\to\xi'(\eta(0))=\alpha^2$, where the derivatives have been calculated from the fixed-point equation. Thus, $c_n(x_0)\to\alpha^2\approx 1.66$ as n tends to infinity.

The asymptotic behavior of $h_n'(x_0)$ is solely determined by those of $c_n(x_0)$ and $d_n(x_0)$. If $h_n'(x_0)$ approached zero slower than $d_n(x_0)$, there would exist an N such that for all $n>N$, $h_{n+1}'(x_0)>Ch_n'(x_0)$ with $C>1$. In this case, $h_n'(x_0)$ would actually keep growing without any limit as $n\to\infty$, which would be contradictory to the tendency of the renormalized circle map to approach the universal function $\eta(x)$. On the other hand, (4) implies that $h_n'(x_0)$ cannot decay to zero faster than $d_n(x_0)$. In other words,

$$h_n'(x_0)=-\frac{d_n(x_0)}{c_n(x_0)}+O(J_{n+1}(x_0))\sim J_n(x_0)\,.$$

It turns out to be possible to work out the limit of the factor

$$e_n(x)=-d_n(x)/c_n(x)J_n(x)$$

at $x=x_0$ approaching the case in which the Jacobian determinant vanishes. Note first that

$$e_2(x)=\frac{g_{12}(h(x))}{g_{12}(x)c_2(x)},\quad e_3(x)=\frac{g_{12}(h_3(x))}{g_{12}(x)c_2(x)c_3(x)}\,.$$

Equation (7) implies a recursion relation for $e_n(x)$, $n=3,4,\ldots$, which becomes very simple in the zero-Jacobian-determinant limit:

$$e_{n+1}(x)\approx e_n(x)e_{n-1}(h_n(x))\,.$$

Because $c_3(x)$ can be replaced by $c_2(h_2(x))c_1(h_2(x))\equiv c_2(h_2(x))$, it is easy to write down the form of a general $e_n(x)$:

$$e_n(x)\approx\frac{g_{12}(h_n(x))}{g_{12}(x)\prod_{i=0}^{F_n-1}c_2(h^i(x))}\approx\frac{1}{\prod_{i=0}^{F_n-1}c_2(h^i(x))}\,.$$

Here I have used the fact that $h_n(x)(\mathrm{mod}\ 1)\to x$ as $n\to\infty$ for the inverse-golden-mean rotation number. Leaving the d term proportional to the Jacobian out of

(4), one obtains

$$\prod_{i=0}^{F_n-1} c_2(h^i(x)) = h_n'(h(x)),$$

where the derivative can be calculated using (8):

$$h_n'(h(x_0+z)) \approx \frac{h'(x_0+\alpha^{-n}\eta(\alpha^n z))\eta'(\alpha^n z)}{h'(x_0+z)}.$$

$\eta(z)$ has a cubic critical point at $z=0$ [8]. Furthermore, also $h(x_0+z)$ develops such a point in the zero-Jacobian-determinant limit. Expanding all the derivatives around $z=0$ and letting $z \to 0$ leads to

$$h_n'(h(x_0)) \to \frac{[\eta(0)]^2\eta'''(0)}{2}.$$

We have thus derived Eq. (1).

Table I shows $h_n'(x_0)/J_n$ for the dissipative standard map,

$$g_1(x,y) = x + \Omega + by - \frac{k}{2\pi}\sin(2\pi x),$$

$$g_2(x,y) = \Omega + by - \frac{k}{2\pi}\sin(2\pi x),$$

with the constant Jacobian determinant $b=0.5$. The critical parameter values for the breakup of the "golden" invariant circle can be determined by a dissipative version of Greene's residue criterion [14]. x_0, y_0 is taken as the point where the approximating periodic orbits have the largest gap. The calculation of $h_n'(x_0)/J_n$ using the forward recursion relation (4) would be extremely sensitive to the choice of the value of $h_2'(x_0) \equiv h'(x_0)$. An error ϵ in $h'(x_0)$ would give rise to an error $\epsilon\prod_{i=2}^{n-1} c_i(x_0)$ in $h_n'(x_0)$ which would be of the order $\epsilon 1.66^{n-2}$. As J_n decays to zero very fast with increasing n ($J_{15} \sim 10^{-184}$), the error in the ratio $h_n'(x_0)/J_n$ would soon become astronomical. A better way to calculate this ratio is to apply (4) backwards beginning with the approximation $h_N'(x_0) \approx 0$ for some large N. The initial error is very small ($\sim J_N$) and the error in $h_n'(x_0)$ is reduced by a factor around 1.66 at each step. In fact, a very good estimate for the ratio $h_n'(x_0)/J_n$ is obtained already for $n=N-1$ if N is not very small. The calculation of $c_n(x_0)$ and $d_n(x_0)$ by (7) appears to be numerically very stable. Thus, all the error arises essentially from the inaccuracy in determining the critical parameter values and the point x_0, y_0.

TABLE I. Subsequent estimates for the factor $h_n'(x_0)/J_n$ resulting from the critical dynamics of the dissipative standard map ($b=0.5$, $k=0.9788377790$, $\Omega=0.3058769514$).

n	$h_n'(x_0)/J_n$
2	0.5280
3	0.4333
4	0.4548
5	0.4146
6	0.4384
7	0.4191
8	0.4385
9	0.4272
10	0.4387
11	0.4312
12	0.4380
13	0.4331
14	0.4373
15	0.4339

Table I shows no deviation from (1) although the system is quite far from the zero-Jacobian-determinant limit. This could be taken as a hint that Eq. (1) would be valid more generically than the derivation would reveal. It would be intriguing to see this tested experimentally. If the experimental data enabled one to construct the one-dimensional return map, it would be quite easy to calculate derivates of higher iterates of this return map by using finite differences. Usually the point x_0 and the Jacobian determinant would not be known. It would be best to estimate the smallest slope of each Fibonacci iterate of the reduced circle map and calculate $h_n'h_{n-1}'/h_{n+1}'$. If the Jacobian determinant varied only little along the invariant circle, this ratio could be close to the universal constant a. It is clear that noise would prevent one from carrying out the calculation for high n. Nevertheless, Table I suggests that even the lowest-order estimate could give a reasonable result.

As to other rotation numbers, I would expect the effective Jacobian again to play an important role [15]. The transition to chaos should be observed by monitoring the smallest slope of the higher iterate $h^{Q_n}(x)$ of the reduced circle map, where Q_n would be the denominator of the nth truncation of the continued-fraction expansion for the rotation number. In the critical case, one would expect this slope to tend to zero as $n \to \infty$, whereas in the subcritical region the asymptotic slope should be unity [8,9].

Equations (4) and (7) can be used to study the conservative case $J(x) \equiv 1$ as well. Both $c_n(x_0)$ and $d_n(x_0)$ have universal nonvanishing limits (x_0 now corresponds to a point on a dominant symmetry line [12]): $c_\infty(x_0) \approx 2.1676633$ and $d_\infty(x_0) \approx -0.4916138$. $c_\infty(x_0)$ appears to be determined by the ratio of the universal phase-space scaling constants [12]. $h_n'(x_0)$ has a universal positive limit $d_\infty(x_0)/[1-c_\infty(x_0)] \approx 0.4210236$. The estimate obtained by setting $J_n=1$ in (1) deviates only about 3% from this true value.

The decimation technique introduced in this Letter is readily applicable also to other problems, e.g., to the discrete quasiperiodic Schrödinger equation [16]. Equation (3) can be interpreted as a discrete eigenvalue equation with h_n' representing the wave vector ψ_{F_n} at site F_n [17]. The modulating potential is included in c_2. Assuming the normalization condition $\psi_0=1$ one can take $d_2 \equiv -1$ The present approach is appropriate when the potential has the frequency ζ relative to the underlying lattice. Infinite products of transfer matrices usually diverge [18], whereas by writing down recursion relations similar to (7), it is possible to find a bounded limiting behavior for the coefficients c_n and d_n [17].

The fact that the slope of the reduced circle map depends on the effective Jacobian is in nice agreement with the conjectured mechanism for the breakup of having a tangency between the invariant circle and its stable foliation [8]. It is natural to think the contraction on the stable foliation is proportional to the effective Jacobian determinant. At the point of tangency, the slope of the associated circle map should therefore be proportional to the Jacobian.

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[1] J. Stavans, F. Heslot, and A. Libchaber, Phys. Rev. Lett. **55**, 596 (1985); A. P. Fein, M. S. Heutmaker, and J. P. Gollub, Phys. Scr. **T9**, 79 (1985); D. J. Olinger and K. R. Sreenivasan, Phys. Rev. Lett. **60**, 797 (1988); R. E. Ecke, R. Mainieri, and T. S. Sullivan, Phys. Rev. A **44**, 8103 (1991).

[2] S. E. Brown, G. Mozurkewich, and G. Grüner, Phys. Rev. Lett. **52**, 2277 (1984); S. Bhattacharya, M. J. Higgins, and J. P. Stokes, Phys. Rev. B **38**, 7177 (1988).

[3] S. Martin and W. Martienssen, Phys. Rev. Lett. **56**, 1522 (1986).

[4] G. A. Held and C. Jeffries, Phys. Rev. Lett. **56**, 1183 (1986); E. G. Gwinn and R. M. Westervelt, *ibid.* **57**, 1060 (1986); A. Cumming and P. S. Linsay, *ibid.* **59**, 1633 (1987); M. Bauer, U. Krueger, and W. Martienssen, Europhys. Lett. **9**, 191 (1989); J. Peinke, J. Parisi, R. P. Huebener, M. Duong-van, and P. Keller, *ibid.* **12**, 13 (1990); W. J. Yeh, D.-R. He, and Y. H. Kao, Phys. Rev. Lett. **52**, 480 (1984); P. Alstrøm and M. T. Levinsen, Phys. Rev. B **31**, 2753 (1985).

[5] J. Maselko and H. L. Swinney, J. Chem. Phys. **85**, 6430 (1986).

[6] T. Bohr, P. Bak, and M. H. Jensen, Phys. Rev. A **30**, 1970 (1984); P. Bak, T. Bohr, M. H. Jensen, and P. V. Christiansen, Solid State Commun. **51**, 231 (1984).

[7] T. Bohr, Phys. Lett. **104A**, 441 (1984).

[8] S. Ostlund, D. Rand, J. Sethna, and E. Siggia, Physica (Amsterdam) **8D**, 303 (1983).

[9] M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, Phys. (Amsterdam) **5D**, 370 (1982).

[10] R. S. MacKay and C. Tresser, Physica (Amsterdam) **19D**, 206 (1986).

[11] X. Wang, R. Mainieri, and J. H. Lowenstein, Phys. Rev. A **40**, 5382 (1989); S.-Y. Kim and B. Hu, *ibid.* **44**, 934 (1991).

[12] R. S. MacKay, Ph.D. thesis, University of Princeton, 1982 (unpublished).

[13] D. Rand, in *New Directions in Dynamical Systems*, edited by T. Bedford and J. Swift (Cambridge Univ. Press, Cambridge, 1988), p. 1.

[14] J. A. Ketoja, Physica (Amsterdam) **55D**, 45 (1992).

[15] The effective Jacobian controls the "universal crossover" in period-doubling systems as well; see C. Reick, Phys. Rev. A **45**, 777 (1992), and references therein.

[16] For a review, see J. B. Sokoloff, Phys. Rep. **126**, 189 (1985).

[17] J. A. Ketoja (to be published).

[18] S. Ostlund and R. Pandit, Phys. Rev. B **29**, 1394 (1984).