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Symmetry Breaking and Localization in Quantum Chaotic Systems

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Using semiclassical arguments substantiated by numerical results for the quantum kicked rotor, we establish that breaking an antiunitary symmetry in a system with dynamical localization increases the localization length by a factor of 2. The transition between the symmetric and the symmetry-broken case is smooth. The semiclassical theory provides an approximate expression for the transition function as well as the critical strength of the symmetry-breaking interaction necessary to achieve the full factor of 2 increase of the localization length.

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Anderson localization was first discovered in the study of electron transport in disordered systems [1]. Later it was also found in certain deterministic Hamiltonian systems, whose classical dynamics is chaotic. A paradigm system is the quantum kicked rotor [2,3] but it occurs also in other systems such as, e.g., the hydrogen atom in a microwave field [4]. These rather simple systems and the disordered (quasi-one-dimensional) solid share three important properties which enable the onset of Anderson localization: (i) the classical phase space is extended, (ii) classical diffusion characterizes the classical evolution, and (iii) many trajectories contribute to the same transition, and the corresponding quantum amplitudes are endowed with phases which are sufficiently uncorrelated to induce Anderson localization as a result of intricate interferences.

It is well known that breaking time-reversal symmetry affects many global properties in a universal way. Well-known examples are spectral statistics and the distribution of transition probabilities and resonance widths [5-9]. The effects of symmetry breaking are commonly studied within random matrix theory [5-9]. It emphasizes the importance of antiunitary symmetries [9] in general, of which an outstanding example is time-reversal symmetry. There is a growing amount of evidence which indicates a strong correlation between localization and symmetry in disordered systems [10,11]. It was shown that in quasi-one-dimensional systems the localization length λ satisfies $\lambda(\beta)=\beta\lambda(\beta=1)$ where $\beta=1$ (or 4) for time-reversal invariant disordered systems which do not (or do) involve spin degrees of freedom. $\beta=2$ for systems with broken time-reversal symmetry.

In the present work we discuss the effects of symmetry breaking on the localization induced by the underlying chaotic dynamics, and compare it to the known behavior of disordered systems. We use semiclassical arguments supported by extensive numerical tests to show that breaking the symmetry modifies the localization length. We will show that the localization length is a continuous function of the strength of the symmetry-breaking interaction. For sufficiently strong interactions the localization length increases by a factor of 2, in accordance with the behavior of disordered systems.

We consider the kicked rotor, whose Hamiltonian is modified to allow for symmetry breaking. We follow the notations introduced in Ref. [12]. We use

$$H=l^2/2+g(\theta;k,q)\sum_{m=-\infty}^{\infty}\delta(t-m\tau)\,, \tag{1}$$

where

$$g(\theta;k,q)=k[\cos(\pi q/2)\cos(\theta)+\tfrac{1}{2}\sin(\pi q/2)\sin(2\theta)]\,.$$

The angular momentum l and the impulse strength k are measured in units of $\hbar$; q is the symmetry-breaking parameter which ranges from 0 to 1 and τ is the time interval between kicks. For $q=0$, (1) reduces to the standard problem whose Hamiltonian is invariant under the following symmetry operations: $T_t: l\to -l, \theta\to\theta, t\to -t$

(time reversal), $T_r: l \rightarrow -l, \theta \rightarrow 2\pi - \theta, t \rightarrow t$ (reflection), and $T_c = T_t \otimes T_r: l \rightarrow l, \theta \rightarrow 2\pi - \theta, t \rightarrow -t$ (conjugation). Both T_t and T_c are antiunitary symmetries. We shall show that the symmetry which is important for the present discussion is T_c. It is broken when $q \neq 0, 1$.

The classical dynamics is chaotic for sufficiently large $K = k\tau$, and the evolution of any initial phase-space distribution is diffusive along the l axis. The form (1) was chosen so as to minimize the effect of symmetry breaking on the classical diffusion constant $D(K,q)$. The ratio $R^{(\mathrm{cl})}(K,q) = D(K,q)/(K^2/2)$ can be approximated by an analytical expression [13]. In Fig. 1 we compare (for $K=20$) the analytical expression (continuous line) and the numerical results (crosses) for $R^{(\mathrm{cl})}(K,q)$. The large deviation between the analytical and numerical results at $q \approx 0.22$ is due to accelerator modes [14]. The remaining deviations are most probably due to breaking the analytical calculations at the third order.

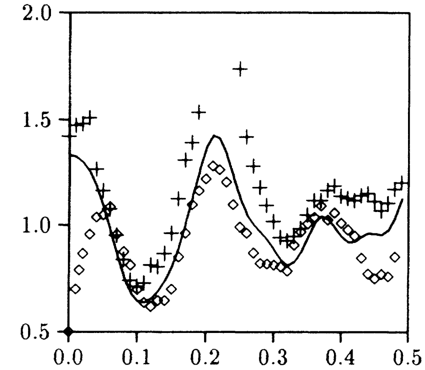

FIG. 1. Solid line: Third-order analytical result approximating the ratio of the classical diffusion constants $R^{(\mathrm{cl})}(K,q)$. Crosses: Monte Carlo result for $R^{(\mathrm{cl})}(K,q)$. Diamonds: $\xi(q)/2\xi(q=0)$ with $\tau = 2\pi \times 0.1/[(\sqrt{5}-1)/2]$ and $k = K/\tau$. In all the cases, $K=20$.

The quantum evolution is determined by the spectral properties of the one-step evolution operator $\hat{U} = \exp(-i\frac{1}{2}\tau\hat{l}^2)\exp[-ig(\hat{\theta};k,q)]$. For τ which is irrationally related to π the spectrum of $\hat{U}$ is believed to be pointlike, with exponentially localized eigenvectors [3]. The q dependence of the localization length of this system is the subject of the present investigation. We checked that the spectrum of a finite version of $\hat{U}(q)$ makes the COE$\rightarrow$CUE (circular orthogonal ensemble to circular unitary ensemble) transition as q gets larger than 0.

A convenient tool for the study of localization is the quantum *mean staying probability* $P_s(n)$, defined as the probability to remain in the "site" l after n kicks, averaged over a large number of l values [12]:

$$P_s(n) = \langle P_{l\rightarrow l}(n)\rangle \equiv \lim_{\Delta\rightarrow\infty} \frac{1}{\Delta} \sum_{l=-\Delta/2}^{\Delta/2} P_{l\rightarrow l}(n), \tag{2}$$

with $P_{l\rightarrow l}(n) = |(\hat{U}^n)_{l,l}|^2$. In the semiclassical limit $P_s(n)$ distinguishes clearly between two distinct time domains [12]: For n smaller than a critical time n^*, the evolution is essentially classical, and $P_s(n)$ is proportional to the classical probability to stay, which for diffusion in one dimension is $(2\pi Dn)^{-1/2}$. The only trace of the quantum evolution appears in the proportionality factor (weak localization) which will be discussed below. The domain $n > n^*$ is dominated by quantum-mechanical localization, and $P_s(n)$ approaches a constant ξ^{-1}—the mean inverse participation ratio. For $q=0$, $P_s(n)$ was found to scale like [12]

$$P_s(n) = \xi^{-1} f(n/\xi). \tag{3}$$

The semiclassical expression for $P_s(n)$ reads

$$P_s^{(\mathrm{sc})}(n) = \left\langle \left| \sum_\alpha p_\alpha^{1/2} \exp(iS_\alpha) \right|^2 \right\rangle, \tag{4}$$

where the sum goes over all classical trajectories (denoted by α) which satisfy $l_\alpha(0) = l_\alpha(n) = l$ (their number proliferates exponentially with n). The action accumulated along the trajectory is S_α. It is measured in units of $\hbar$, and it includes the Maslov index. p_α is the contribution of the trajectory α to the classical probability, $p_\alpha = (1/2\pi)|dl_\alpha(n)/d\theta_\alpha(0)|^{-1}$.

If the Hamiltonian is invariant under a symmetry operation *which does not affect the boundary conditions* $l_\alpha(0) = l_\alpha(n) = l$, we can assign a conjugate trajectory α_c to any trajectory α by applying the symmetry operation to the trajectory. One should consider only antiunitary symmetries. Unitary symmetries imply reducibility of the dynamics and their effects can be taken into account by working in an appropriate representation. This is not the case when the symmetry is antiunitary. Among the symmetry operations listed above, only T_c provides a relevant conjugation, since this is the only antiunitary symmetry which preserves the sign of l. For $q=0$, conjugate trajectories contribute equal amplitudes to the sum on α in the semiclassical expression (4). For small q, one can still use the conjugation symmetry to identify pairs of trajectories which become a symmetric pair for $q \rightarrow 0$. One can approximate $p_\alpha(q) \approx p_{\alpha_c}(q) \approx p_\alpha(q=0)$, but one must take note of the change in the actions. To leading order in q the action differences $\delta_\alpha(q;n) \equiv S_\alpha(q) - S_\alpha(0) = -[S_{\alpha_c}(q) - S_{\alpha_c}(0)]$ are given by

$$\delta_\alpha(q;n) = \tfrac{1}{2} k \sin\left(\frac{\pi q}{2}\right) \sum_{i=1}^{n} \sin[2\theta_\alpha(i)], \tag{5}$$

where the α trajectory at $q=0$ is $[l_\alpha(i), \theta_\alpha(i)]_{i=1}^{n}$.

The mean staying probability can now be written as

$$P_s^{(\mathrm{sc})}(n;q) = 4\left\langle \sum_\alpha{}' p_\alpha \cos^2\delta_\alpha(q;n) \right\rangle + 4\left\langle \sum_{\alpha\neq\beta}{}' (p_\alpha p_\beta)^{1/2} \exp[i(S_\alpha - S_\beta)] \cos\delta_\alpha \cos\delta_\beta \right\rangle. \tag{6}$$

The primed summation symbol indicates summation over pairs rather than individual trajectories. The first term in (6) (the "diagonal" term) can be viewed as the averaged value of $\cos^2\delta_\alpha$ with weights p_α over the ensemble of all trajectories α which contribute to the amplitude to stay in l after n kicks. The sum over all weights p_α adds up to the classical probability to stay, $P_s^{(cl)}(n;q)$. Hence, the diagonal contribution can be written as

$$P_s^{(D)}(n;q)=\left\langle 2{\sum_\alpha}' \tilde{p}_\alpha \cos^2\delta_\alpha(n;q)\right\rangle P_s^{(cl)}(n;q) \tag{7}$$

with $\tilde{p}_\alpha=2p_\alpha/P_s^{(cl)}$ and $\sum'\tilde{p}_\alpha=1$. According to (5) the phases δ_α are a sum of n terms. For chaotic systems, the successive $\sin\theta_\alpha(i)$ can be considered statistically independent. Therefore, for large n, the phases δ_α form a Gaussian ensemble with vanishing mean and with variance $\langle\delta_\alpha^2\rangle=\frac{1}{2}n[\frac{1}{2}k\sin(\frac{1}{2}\pi q)]^2$. For $n<n^*$ the probability to stay is entirely dominated by the diagonal contribution, since the nondiagonal term vanishes upon averaging. Hence,

$$\begin{aligned}P_s(n;q)&=P_s^{(cl)}(n)\left\{1+\exp\left[-n\frac{k^2}{4}\sin^2\left(\frac{\pi q}{2}\right)\right]\right\}\\&=P_s(n;q=0)\left\{1+\exp\left[-n\frac{k^2}{4}\sin^2\left(\frac{\pi q}{2}\right)\right]\right\}\Big/2.\end{aligned} \tag{8}$$

This expression is valid only in the "diffusive" domain. It shows that the quantal probability differs from its classical counterpart by a smooth function of time which depends on the symmetry-breaking parameter q in a way which interpolates smoothly between the limits 2 (for $q=0$) and 1 for large q. The factor 2 is the typical "weak localization" enhancement which is known in many fields of physics and is due to the invariance of the system under a symmetry of the type discussed above [15-17]. The semiclassical derivation predicts how the symmetry-breaking interaction erases the weak localization signature, in excellent agreement with the numerical simulations (see Fig. 2). If we replace in $g(\theta;k,q)$ the term $\sin(2\theta)$ by $\cos(2\theta)$, we expect $P_s(n;q)=P_s(n;q=0)$ since no symmetry is broken. This is confirmed by the numerical results (dashed ragged line in Fig. 2).

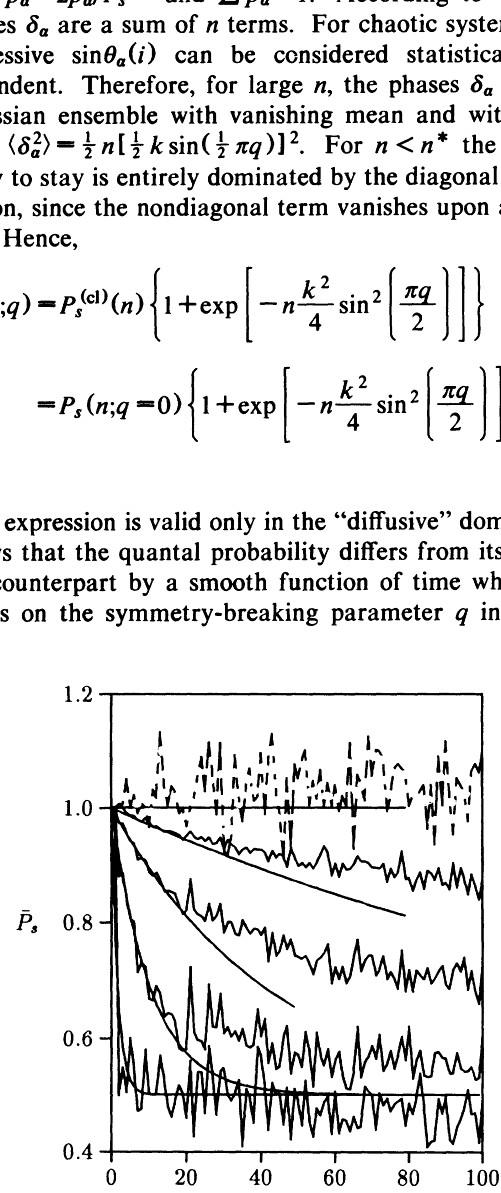

FIG. 2. Ragged lines: Normalized quantum staying probabilities $\bar{P}_s=P_s(n,q)/P_s(n,q=0)$ for $q=0.005$, 0.01, 0.02, 0.05. Smooth lines: The semiclassical prediction (8). The dashed ragged line is for an interaction which does not break the symmetry (see text).

The diagonal contribution to the staying probability diminishes as $n^{-1/2}$. Hence, the long-time properties of $P_s(n)$ are due exclusively to the nondiagonal term in (6). This is the semiclassical manifestation of the well-known fact that localization [here, finite value of $P_s(n)$, $n>n^*$] is due to genuine interference effects. The evaluation of such nondiagonal sums is one of the less understood issues in the semiclassical treatment of classically chaotic systems [18]. They involve the pair correlations of classical actions about which very little is known. Thus, for the discussion of the time domain where quantum effects dominate, we must resort to numerical studies.

When $\frac{1}{2}k\sin(\pi q/2)>1$ we are in the extreme broken-symmetry regime where $P_s(n,q)=P_s(n,0)/2$ for all $n<n^*$. We found numerically that this relation holds also in the $n>n^*$ regime, which implies that $\xi(\beta)=\beta\xi(\beta=1)$. This is equivalent to what is known from the random matrix treatment of localization in quasi-one-dimensional disordered systems. Our numerical data suggest an even stronger result, namely, that in the symmetry-broken regime, the function $P_s(n;\beta)$ scales as

$$P_s(n;\beta)=\frac{1}{\xi(\beta)}f\left(\frac{\beta n}{\xi(\beta)}\right), \tag{9}$$

where the function $f(x)$ is the same function as defined for the time-reversal symmetric case (see Fig. 3). This is a rather surprising result since $f(t)$ is the Fourier transform of the two-point cluster function for the local quasienergy spectrum [12]. Equation (9) shows that but

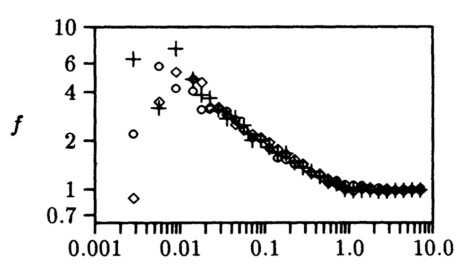

FIG. 3. Scaled staying probability $f(x)=\xi(q)P_s(q,n)$ as a function of the scaled time $x=\beta n/\xi(q=0)$ for $q=0$ ($\beta=1$, circles) and $q=0.32$, 0.5 ($\beta=2$, crosses and diamonds, respectively). $k=20/\tau$, $\tau=2\pi\times0.05/[(\sqrt{5}-1)/2]$.

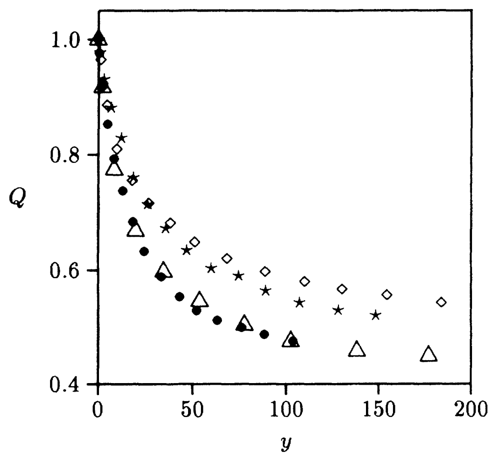

FIG. 4. Check of the scaling behavior (10) for various values of q and $k=23.42$, 19.67, 17.88, 16.39 for triangles, diamonds, stars, and bullets, respectively.

for scaling, the *same* function applies in the $\beta=1$ or 2 ensembles. Such a relation does not hold in the corresponding Dyson ensembles.

A prerequisite for any scaling relation of the type (9) is that the participation ratio and the classical diffusion constants are proportional. In Fig. 1 we show in diamonds the ratio $\xi(q)/2\xi(q=0)$. We see that this ratio is close to unity on average, and that it oscillates in a way which is rather similar to the corresponding ratio of the classical diffusion constants.

Lacking the analytical tools to evaluate the probability to stay for $n>n^*$, we are not able to discuss the smooth transition of ξ between the extreme domains. Our numerical work suggests that to a good approximation the time-averaged $P_s(n;q)$ $(n>n^*)$, or equivalently $1/\xi(q)$, is given by

$$\frac{1}{\xi(q)} \approx \frac{1}{\xi(q=0)}\left\{1+\exp\left[-n^*\frac{k^2}{4}\sin^2\left(\frac{\pi q}{2}\right)\right]\right\}\Big/2. \tag{10}$$

Since $n^*\sim D$, (10) predicts that the ratio $Q(q)=\xi(q=0)/\xi(q)$ scales in $y\equiv D(k,q)k^2\sin^2(\pi q/2)$. Figure 4 shows that overall our data reflect this scaling to a good accuracy. The origin of the deviations is presently not understood.

In summary, our numerical results clearly show that the localization length depends smoothly on the strength of the symmetry-breaking interaction. For completely broken symmetry, the participation ratio is twice as large as its value for the symmetric case. This behavior is analogous to the theoretical results for quasi-one-dimensional disordered systems. Our work shows that the critical-field strength needed to break the symmetry can be evaluated by requiring that the rms of the action due to the symmetry-breaking force acting during the localization time $n^*\approx\xi$ is of order $\hbar$. The study of the role of symmetry breaking in systems of higher dimension is still an open problem.

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Note added.—Recently we checked the case of symplectic symmetry by coupling the rotor to a spin degree of freedom. We found that all the scaling relations also hold for $\beta=4$.

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