Universal Properties of the Wave Functions of Fractional Quantum Hall Systems

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We show that the wave functions of the fluid ground states of fractional quantum Hall systems, in the thermodynamic limit, are universal at long distances and that they have a generalized Laughlin form. This universality is a consequence of the analytic properties of the equal-time density correlation functions at long distances. The correlation functions calculated from the field theoretic approach to the fractional quantum Hall effect have the correct analytic properties and the wave function calculated in the Gaussian approximation becomes exact in the asymptotic limit.

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The Laughlin wave function for the fractional quantum Hall effect (FQHE) is a fascinating object. It describes the behavior of interacting electrons in a heterostructure in the presence of a strong magnetic field. This wave function has an exceedingly simple structure which does not depend on properties of the material and has almost no dependence on the magnetic field itself. For an interacting two-dimensional electron gas (2DEG) in a perpendicular field of strength B and filling fraction $\nu=1/m$ (m an odd integer), the Laughlin wave function [1] is

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^m \exp\left(-\sum_{i=1}^{N}\frac{|z_i|^2}{4l^2}\right). \quad (1)$$

In this sense, it is *universal*. But, what is the physical origin of its universality? Wave functions of many-body systems are usually very complex objects. At some level of approximation they can have simple intuitive properties, e.g., the BCS wave function for superconductors. However, in general, they are neither simple nor universal. A number of arguments have been given to explain this universality. Laughlin [1] and Haldane [2] have argued that this is the unique Jastrow wave function, antisymmetric in the coordinates, with a given value of the angular momentum that can be written entirely in terms of the single particle states of only the lowest Landau level. Thus, as the Landau gap is taken to infinity, this should be *the* wave function. Trugman and Kivelson [3] and Haldane [2] have constructed ultralocal model Hamiltonians which, by construction, have the Laughlin wave function as the ground state. There has also been significant amount of effort to treat the FQHE problem within a conventional field theoretic approach [4–7]. In these schemes, the physical properties of the system are derived without the explicit use of the wave function. It is hard to understand what the relationship is between these two approaches. This motivates the issue of deriving the wave function from the field theory as a way of understanding the origin of its universality. Recently Kane *et al.* [8,9] showed that the wave function obtained from the Landau-Ginzburg field theory of FQHE coincides, in the Gaussian approximation, with the Laughlin wave function. Using methods of collective field theory, Sakita [10] has also proposed a derivation of the Laughlin wave function. But, in more complex FQHE states which require the use of the hierarchy [11–13], no such simple construction is yet available. However, it is not clear why the Gaussian approximation to the field theory should give exactly the Laughlin wave function. This result suggests that there should be a limit in which the Gaussian approximation is exact [14].

In this paper we show that, in the thermodynamic limit, the exact asymptotic properties of the wave function, when its arguments are separated by distances long compared with the cyclotron length (but short compared with the linear size of the system), are completely determined by the long-distance behavior of the equal-time density-density correlation function (i.e., the structure factor). We show that the universality of the wave function in this limit is a direct consequence of the gauge invariance and of the analytic properties of the electromagnetic response functions which, in turn, follow from incompressibility and Galilean (magnetic) invariance. We explicitly show that the absolute value of the exact wave function for the incompressible ground states has the same form, at long distances, as the Laughlin wave function with a universal exponent equal to $2/\nu$. This result implies that, in general and for systems in the thermodynamic limit, the asymptotic form of the wave function is necessarily nonanalytic since $2/\nu$ is not generally equal to an even integer. It predicts the well-known results for a single Landau level and for the states in the Laughlin sequence. It also predicts a nonanalytic behavior for the states in the hierarchies [11–13] and for IQHE states with $\nu>1$. This singular behavior is valid when any pair of variables of the wave function $\mathbf{x}_j$ and $\mathbf{x}_k$ satisfies $\max(l,d)\ll|\mathbf{x}_j-\mathbf{x}_k|\ll R$, where d is the mean particle separation and R is the linear size of the system. This singular behavior implies that, *in the thermodynamic limit*, the wave functions for states of the hierarchy, such as $\nu=\frac{2}{5}$, cannot be written only in terms of purely analytic functions. The analytic form of both the hierarchical and Jain wave functions is so involved that explicit formulas valid in the thermodynamic limit are not available. Even the form for the "simple" case $\nu=2$ is not known. This result is applicable to a broad class of models of a two-dimensional electron gas, with reasonably local interactions and with incompressible ground states [15]. In particular, the fermion field theory of Ref. [6] (hereafter referred to as I), as well as the Landau-Ginzburg approach [9], yields the response functions in the form of an expansion in powers of $1/B$, which satisfy the required symmetry and analyticity properties. Therefore, the asymptotic form of the wave function of these theories coincides with the Laughlin state [16].

We begin with the problem of the calculation of the wave function from field theory. In order to write down the wave function for this system, we have to choose a complete set of eigenstates on which to project the ground state. In the absence of external sources, the 2DEG conserves the electric charge. The charge and current operators $\hat{\rho}(\mathbf{x})$ and $\hat{J}_k(\mathbf{x})$ $(k=1,2)$ obey a continuity equation and the system has a local electromagnetic gauge invariance. The charge and current operators satisfy an algebra which follows from the anticommutation relations of the electron field and from gauge invariance [17,18]. Therefore, the current and the density operators do not commute with each other. Since a perpendicular magnetic field is present, the commutator of the space components of the current at equal times is nonzero. Hence, the eigenstates of the 2DEG in the presence of a perpendicular magnetic field are representations of the group of magnetic translations (or magnetic group). We will work in the *density representation* [19]. The eigenstates of the density operator, given by $\sum_{i=1}^{N}\delta(\mathbf{x}-\mathbf{x}_i)-\bar{\rho}$ $(\bar{\rho}=N/L^2)$, form a basis of the subspace of states with a fixed number of particles.

In Ref. [20], one of us derived a formula which gives the absolute value squared of the wave function of the ground state of a field theory in terms of the generating function of the equal-time correlation functions. Using the same method we write the square modulus of the ground-state wave function for a fractional quantum Hall system, in the density representation, in the form

$$|\Psi[\rho]|^2=\int \mathcal{D}A_0\langle \mathrm{GS}|Te^{-i\int A_0\hat{\rho}}|\mathrm{GS}\rangle e^{i\int A_0\rho}\,, \qquad (2)$$

where $|\mathrm{GS}\rangle$ is the ground state. Since we are interested only in equal-time correlations, the time component of the electromagnetic field A_0 must act only at one time x_0, i.e., $A_0(x)=A_0(\mathbf{x})\delta(x_0)$. The expression in the integrand of Eq. (2) is

$$\langle \mathrm{GS}|Te^{-i\int A_0\hat{\rho}}|\mathrm{GS}\rangle=\int \mathcal{D}\psi^\dagger\mathcal{D}\psi\mathcal{D}a_\mu \times\exp\{iS(\psi^\dagger,\psi,a_\mu,A_\mu)\}\,. \qquad (3)$$

Here ψ are the fermionic fields and a_μ is the gauge field introduced in I. For arbitrary values of an external weak electromagnetic field A_μ, the path integral of the right-hand side of Eq. (3) is the generating function of time-ordered density and current Green functions. The equal-time density correlation functions are obtained from Eq. (3) by setting $A_0(x)=A_0(\mathbf{x})\delta(x_0)$ and all the space components of the external electromagnetic field to zero. Notice that this refers only to the weak perturbation, not to the fixed magnetic field B.

In paper I, we presented a theory of the FQHE based on the method of second quantization with fermion path integrals. We showed that there is an exact equivalence between the interacting 2DEG and the same system coupled to a Chern-Simons vector potential a_μ, the statistical vector potential, whose only role is to bind each electron to an even number of flux quanta. This theory has a semiclassical (or saddle-point) expansion in powers of $\hbar$. Because of the macroscopic degeneracy of the Landau levels, the semiclassical expansion becomes an expansion in powers of the inverse of the external magnetic field $1/B$ [21]. But, the gradient expansion is also an expansion in powers of $1/B$. This fact has important consequences in what follows below. The leading order of this expansion is the *average field approximation* (AFA). We found that, at the level of the AFA, the system has a gap if the filling fraction ν satisfies $1/\nu=1/p+2s$, where p and s are arbitrary positive integers [22]. The case $p=1$ is the Laughlin sequence. It is the existence of a nonzero gap at the level of the AFA that makes the approximation consistent. We found that the Gaussian (or semiclassical) fluctuations of the statistical vector potential are responsible for the FQHE in the sense that, at the semiclassical level, the Hall conductance has exactly the correct value. One of us [23] has also showed that the quantum numbers of the quasiparticles (and quasiholes) are exact at the Gaussian level. In Ref. [7] (hereafter called II), we have also calculated the semiclassical approximation to the linear response functions of the system to small external electromagnetic perturbations. The following results from II are crucial to the derivation of asymptotic properties of the wave function: (a) To leading order in $\mathbf{Q}^2$ the electromagnetic response functions have a pole at the Kohn mode [24] with weight $\mathbf{Q}^2$, and (b) as a consequence of gauge invariance the higher-order response functions have higher powers in $\mathbf{Q}^2$ and, hence, in $1/B$. These results hold for any model Hamiltonian of the 2DEG with reasonably local interactions, i.e., with pair interactions which obey $\mathbf{Q}^2\tilde{V}(Q)\to 0$ as $\mathbf{Q}^2\to 0$, where $\tilde{V}(Q)$ is the Fourier transform of the interaction potential.

The path integral on the right-hand side of Eq. (3) can be written in terms of the effective action $S_{\mathrm{eff}}(A_\mu)$ for the external electromagnetic field. In the thermodynamic limit, and for weak fields, the effective action admits the expansion [7]

$$S_{\mathrm{eff}}(A_\mu)=\frac{1}{2}\int d^3x\int d^3y\,A_\mu(x)K^{\mu\nu}(x,y)A_\nu(y)+\cdots\,. \qquad (4)$$

Here $K^{\mu\nu}$ is the polarization tensor. Since we need only the density correlation functions, it suffices to know the zero-zero component of $K^{\mu\nu}$. In momentum space, and in the small $\mathbf{Q}^2$ limit, we find [7]

$$K^{00}(\omega,\mathbf{Q})=-\frac{\bar{\rho}}{M}\frac{\mathbf{Q}^2}{\omega^2-\omega_c^2+i\epsilon}+\mathcal{O}((\mathbf{Q}^2)^2). \quad (5)$$

The dominant term in K^{00} is of order $1/B$. Higher-order terms in the gradient expansion will contribute with higher powers of $1/B$. The same observation applies for all the corrections to K^{00} originating in higher-order terms in the semiclassical expansion. Here the thermodynamic limit is crucial since we are only taking into account fluctuations with wavelengths short compared with the linear size of the system. The higher-order terms, which vanish like powers of $\mathbf{Q}^2/B$, can only be neglected for an infinite system.

An important feature of this result [Eq. (5)] is that *it saturates the f-sum rule.* In units in which $e=c=\hbar=1$, the sum rule for the retarded density-density correlation function D^R_{00} states that

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}i\omega D^R_{00}(\omega,\mathbf{Q})=\frac{\bar{\rho}}{M}\mathbf{Q}^2. \quad (6)$$

Using the relation between the polarization tensor and the current-current correlation functions (see, for example, Ref. [23]), and the relation between retarded and time-ordered Green's functions, we can verify that the *leading-order term* of K_{00} saturates the sum rule. Thus, higher-order corrections in the expansion cannot modify this result.

Using Eqs. (4) and (5), Eq. (3) becomes

$$\langle \mathrm{GS}|Te^{-i\int A_0\hat{\rho}}|\mathrm{GS}\rangle=\exp\left\{\frac{i}{2}\int d^2x\int d^2y\,A_0(\mathbf{x})\left(\lim_{x_0\to y_0}K^{00}(x,y)\right)A_0(\mathbf{y})+\cdots\right\}, \quad (7)$$

or, by Fourier transforming the exponent, we get

$$\langle \mathrm{GS}|Te^{-i\int A_0\hat{\rho}}|\mathrm{GS}\rangle=\exp\left\{\frac{i}{2}\int\frac{d^2Q}{(2\pi)^2}A_0(\mathbf{Q})\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}K^{00}(\omega,\mathbf{Q})\right]A_0(-\mathbf{Q})+\cdots\right\}. \quad (8)$$

As we remarked before, the leading-order term in $\mathbf{Q}^2$ in K^{00} saturates the sum rule. Hence, none of the higher-order terms in the gradient expansion (or $\mathbf{Q}^2$ expansion) or in the semiclassical expansion of the path integral can possibly modify the coefficient of the $\mathbf{Q}^2$ term in the exponent of the right-hand side of Eq. (8). In other words, the corrections coming from non-Gaussian and short-distance fluctuations cannot modify the leading-order behavior of K^{00}, and, therefore, the leading-order behavior at long distances of the wave function.

Furthermore, the terms dropped in the exponent of Eqs. (7) and (8) represent equal-time density correlation functions with more than two densities. These terms give rise to three-body corrections (and higher) to the wave function and modify the Jastrow form. The kernels of these nonlinear contributions are, by gauge invariance, required to be transverse. Thus, in momentum space, the residues of their poles have higher powers in $\mathbf{Q}^2$ than $K^{00}(\omega,\mathbf{Q})$. Since, by dimensional analysis, each power of $\mathbf{Q}^2$ has to come with a factor of $1/B$, these terms which are not bilinear in the densities are subleading contributions in the limit $B\to\infty$. At the level of the Gaussian (or semiclassical) approximation, these kernels are equal to zero. All of these considerations hold provided that the Fourier transform of the pair potential satisfies $\mathbf{Q}^2\tilde{V}(Q)\to 0$ as $\mathbf{Q}^2\to 0$.

Thus, we find that the absolute value squared of the wave function is

$$|\Psi[\rho]|^2=\int\mathcal{D}A_0\exp\left\{-\int\frac{d^2Q}{(2\pi)^2}A_0(\mathbf{Q})\frac{\mathbf{Q}^2\nu}{8\pi}A_0(-\mathbf{Q})-i\int\frac{d^2Q}{(2\pi)^2}A_0(\mathbf{Q})\rho(-\mathbf{Q})\right\}. \quad (9)$$

To obtain this result we used that $\bar{\rho}/B=\nu/2\pi$ and that $\omega_c=B/M$. Since the exponent of the integrand is quadratic in A_0, the functional integral of Eq. (9) can be calculated explicitly. After transforming back into real space we get

$$|\Psi[\rho]|^2=\exp\left\{\frac{1}{\nu}\int d^2x\int d^2y\,\rho(\mathbf{x})\ln\left[\frac{|\mathbf{x}-\mathbf{y}|}{R}\right]\rho(\mathbf{y})\right\}. \quad (10)$$

Here R is the radius of the droplet which serves as a long-distance cutoff. In a subspace with N particles, the density eigenvalues are $\rho(\mathbf{x})=\sum_{i=1}^N\delta(\mathbf{x}-\mathbf{x}_i)-\bar{\rho}$. Thus, we get

$$|\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)|^2=\exp\left\{\frac{2}{\nu}\sum_{i<j=1}^N\ln|\mathbf{x}_i-\mathbf{x}_j|-\frac{B}{2}\sum_{i=1}^N|\mathbf{x}_i|^2\right\}, \quad (11)$$

or, equivalently,

$$|\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)|^2=\prod_{i<j=1}^N|\mathbf{x}_i-\mathbf{x}_j|^{2/\nu}\exp\left\{-\frac{B}{2}\sum_{i=1}^N|\mathbf{x}_i|^2\right\}. \quad (12)$$

Equation (12) stands as a generalization of the plasma analogy, valid only at long distances.

The arguments which lead to the universal form of Eq.

(12) are the consequence of the following facts: (a) The natural expansion parameter for this problem is $1/B$, (b) each correction to the response functions beyond the linear order is, by dimensional analysis and gauge invariance, of order $\mathbf{Q}^2/B$, and (c) the density-density correlation function at equal times has a pole at the cyclotron mode (in accordance with Kohn's theorem) with a residue of order $\mathbf{Q}^2$ (required by gauge invariance) with a coefficient which saturates the f-sum rule. These arguments do not rely on a specific model calculation and are completely general. However, the $1/B$ expansion is only consistent for incompressible states [15]. We have further verified that the field theory introduced in I satisfies these requirements already at the semiclassical level. The universal form of Eq. (12) holds for all the FQHE states in the sequence $1/\nu = 1/p + 2s$, including, of course, the Laughlin states. However, except for the states in the Laughlin sequence $1/\nu = 2s+1$, the factors in Eq. (12) have powers which are not even integers. Hence, in the general case, it is not possible to find a set of phases (i.e., a gauge) in which the wave function is holomorphic (up to the exponential factors). This means that the wave functions for the more general states in that sequence cannot be written only in terms of single particle states of the lowest Landau level, even in the limit $B \to \infty$ (we will elaborate on this issue in [7]). The arguments that we have used for the ground state tell us that the wave functions of the excited states also have a universal long-distance limit. Note that what we proved here is the universality of the FQHE states at distances long compared with the cyclotron radius l. At length scales comparable with l nonuniversal short-distance effects become important. Details of the electron-electron interaction and of the properties of the material determine the actual energy of the state. At these length scales the state does not necessarily have the Laughlin form. Yet, our semiclassical approach is a viable scheme for the study of these effects.

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[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[2] F. D. M. Haldane, in *The Quantum Hall Effect*, edited by R. Prange and S. Girvin (Springer-Verlag, New York, 1989).

[3] S. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985).

[4] S. C. Zhang, T. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[5] N. Read, Phys. Rev. Lett. **62**, 86 (1989).

[6] A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991).

[7] Ana Lopez and Eduardo Fradkin (to be published).

[8] C. Kane, S. Kivelson, D. H. Lee, and S. C. Zhang, Phys. Rev. B **43**, 3255 (1991); a qualitative but related argument was also given in D. H. Lee, G. Baskaran, and S. Kivelson, Phys. Rev. Lett. **59**, 2467 (1987).

[9] S. C. Zhang, Int. J. Mod. Phys. B **6**, 25 (1992).

[10] D. N. Sheng, Z. B. Su, and B. Sakita, Int. J. Mod. Phys. B **5**, 1715 (1991).

[11] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[12] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[13] J. Jain, Phys. Rev. B **40**, 8079 (1989).

[14] Girvin has argued, in the context of spin liquids, that there is a relation between the form of wave functions of the Laughlin type and the behavior of the structure factor at long wavelengths; S. Girvin, in *The Quantum Hall Effect* (Ref. [2]), Appendix.

[15] The incompressible states have a gap which scales with B and, thus, $1/B$ (or the inverse of the gap) is a natural expansion parameter. This expansion is well behaved at long wavelengths. But, if the state is compressible, the state is gapless and the $1/B$ expansion has infrared divergences at zero temperature. This behavior signals a breakdown of the expansion and, perhaps, an instability of the *compressible* fluid state.

[16] See also the remarks following Eq. (12).

[17] L. P. Kadanoff and P. C. Martin, Phys. Rev. **124**, 677 (1961).

[18] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).

[19] We could also choose a representation in which a linear combination of the space components of the current is diagonal. In the presence of the magnetic field, it is not possible to find an eigenstate which is simultaneously an eigenstate of the two components of the current.

[20] E. Fradkin, University of Illinois at Urbana-Champaign report, 1992 (to be published).

[21] This applies to the fluctuations of the statistical vector potential a_μ in the FQHE regime. In contrast, in the integer regime, the expansion parameter is $|V|/B$.

[22] This result was also found by J. Jain (Ref. [13]), and by G. Greiter and F. Wilczek, *Fractional Statistics and Anyon Superconductivity* by F. Wilczek (World Scientific, Singapore, 1991).

[23] E. Fradkin *Field Theories of Condensed Matter Systems* (Addison-Wesley, Redwood City, CA, 1991).

[24] Walter Kohn, Phys. Rev. **123**, 1242 (1961).